Art. I.—*On Practical Geodesy.*

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The method of investigation employed in this paper is of a purely elementary character, and in this respect it differs from that usually adopted by the most distinguished geometers who have written on the subject. The method introduced by Legendre, Delambre, and Puissant, and which has been followed by Airy and others, is characterised chiefly by the subsidiary use of the higher calculus and interminable series.

The elementary method here pursued leads to simpler and more comprehensive formulæ, and at the same time affords a clearer insight into the various relations between latitudes, azimuths, differences of longitude, length and circular measure of geodesic arc, angles of depression of the chord, &c. Its power of improving and extending the science in one of its most useful directions can be judged of from the numerous new results arrived at, and a comparison between them and those hitherto evolved by means of the higher calculus.

The errors which have been shewn to exist in some of the investigations and formulæ given in the "account" of the principal triangulation of Great Britain and Ireland, will no doubt attract the attention of Engineers and Surveyors engaged on trigonometrical surveys in India and elsewhere.

Let P_o be the pole of reference of the spheroidal earth;
„ C_o be the centre of the earth;
„ S_o, S_{oo}, be any two stations on the earth's surface;
„ Z_o, Z_{oo}, be the points in which the normals at the respective stations S_o, S_{oo}, cut the earth's polar axis.

The planes $S_oZ_oS_{oo}$, $S_{oo}Z_{oo}S_o$, are "the normal-chordal planes." And any plane whatever which contains the chord of the geodesic arc S_oS_{oo} shall be referred to as a chordal plane.

The polar and equatorial radii of the earth being 20855233, and 20926348 feet, it is easy to show that for arcs on its surface not more than 528000 feet or 100 miles in length, we may consider the traces of the two normal-chordal planes as equals in length and circular measure to that of the "true geodesic" or shortest arc between the stations.

Conceive two unit spheres described, having S_o, S_{oo}, as centres. Let $C_{\prime}$, $S_{\prime}$, I, P, be the points in which the sphere S_o is pierced by the productions of the lines C_oS_o, Z_oS_o, $S_{oo}S_o$, through the centre S_o, and by the line S_oP parallel to and in the same direction as the polar axis C_oP_o.

Let $C_{\prime\prime}$, $S_{\prime\prime}$, $I_{\prime\prime}$, $P_{\prime\prime}$, be the points in which the sphere S_{oo} is pierced by the productions of the lines C_oS_{oo}, $Z_{oo}S_{oo}$, by the chord S_oS_{oo} taken in the direction $S_{oo}S_o$, and by the line $S_{oo}P_{\prime\prime}$ parallel to and in the same direction as the polar radius C_oP_o.

Then evidently the points P, C, $S_{\prime}$, are in the trace, on the unit sphere S_o, of the earth's meridian plane through S_o; and $P_{\prime\prime}$, $C_{\prime\prime}$, $S_{\prime\prime}$, are in the trace, on the unit sphere S_{oo}, of the earth's meridian plane through the station S_{oo}.

The arc $P_{\prime\prime}I_{\prime\prime}$ is equal to the arc PI, each of them being the measure of the angle which the chord joining the stations makes with the earth's polar axis.

The angle $P_{\prime\prime}S_{\prime\prime}I_{\prime\prime}$ is the azimuth of the station S_o as observed at the station S_{oo}; and the angle $PS_{\prime}I$ is the supplement of the azimuth of the station S_{oo} as observed at the station S_o. The arcs $PS_{\prime}$, $P_{\prime\prime}S_{\prime\prime}$, are the geographic colatitudes of the stations S_oS_{oo},—such as can be measured directly by means of the Zenith Sector.

The arcs $PC_{\prime}$, $PC_{\prime\prime}$, are the geocentric colatitudes of the stations.

Now conceive the unit sphere S_{oo} moved by direct translation along the chord, carrying its lines and points rigidly fixed, until its centre coincides with the centre S_o of the unit sphere S_o. It is evident that the points $I_{\prime\prime}$, $P_{\prime\prime}$, will coincide with I, P, and that the points I, $C_{\prime}$, $C_{\prime\prime}$, lie in one great circle of the sphere S_o. It is also evident that the points $P_{\prime}$, $S_{\prime\prime}$, $C_{\prime\prime}$, lie in one great circle of the unit sphere S_o, and that the spherical angle $S_{\prime}PS_{\prime\prime}$ or $C_{\prime}PC_{\prime\prime}$ is equivalent to the difference of longitude of the stations S_oS_{oo}.

Let $p_{\prime}$, $p_{\prime\prime}$, be the points in which the lines PS_o, $P_{\prime\prime}S_{oo}$, parallel to the polar axis, pierce the earth's equator. Then it is evident that the plane angle $p_{\prime}C_{\circ}p_{\prime\prime}$ is equivalent to the difference of longitude of the stations.

It is also evident that the plane angles $C_{\circ}p_{\prime}p_{\prime\prime}$, $C_{\circ}p_{\prime\prime}p_{\prime}$ are equals respectively to the spherical angle $S_{\prime}PI$, and the supplement of the spherical angle $S_{\prime\prime}PI$.

Let $D_{\prime}$, $D_{\prime\prime}$ be the points in which the great circles $IS_{\prime\prime}$, $IS_{\prime}$ cut the great circles $PS_{\prime}C_{\prime}$, $PS_{\prime\prime}C_{\prime\prime}$, respectively. It is evident the arc $S_{\prime}S_{\prime\prime}$ is the measure of the angle which the normals make with each other.

The arc $S_{\prime}D_{\prime\prime}$ is the measure of the plane angle $S_{\circ}Z_{\circ}S_{\circ\circ}$; the arc $S_{\prime\prime}D_{\prime}$ is the measure of the plane angle $S_{\circ\circ}Z_{\circ\circ}S_{\circ}$; the arcs $S_{\prime}C_{\prime}$, $S_{\prime\prime}C_{\prime\prime}$ are the measures of "the angles of the vertical" at the stations $S_{\circ}S_{\circ\circ}$; the spherical angle $S_{\prime}IS_{\prime\prime}$ is equal to the angle between the two normal-chordal planes.

And if O, $E_{\prime}$, $E_{\prime\prime}$ be the points in which the great circle of the unit sphere having I as pole cuts the arcs $S_{\prime}S_{\prime\prime}$, $S_{\prime}D_{\prime\prime}$, $S_{\prime\prime}D_{\prime}$ respectively; it is evident that the arcs $S_{\prime}E_{\prime}$, $S_{\prime\prime}E_{\prime\prime}$ are the measures of the angles of depression of the geodesic chord $S_{\circ}S_{\circ\circ}$ below the tangent planes to the spheroidal earth at the respective stations $S_{\circ}S_{\circ\circ}$; and they are the complements of the angles which the normals make with the chord.

The spherical angles $S_{\prime\prime}S_{\prime}D_{\prime\prime}$, $S_{\prime}S_{\prime\prime}D_{\prime}$ are equivalents to the angles which any plane parallel to the two normals makes with the two normal-chordal planes.

And the spherical angles $S_{\prime\prime}D_{\prime}D_{\prime\prime}$, $S_{\prime}D_{\prime\prime}D_{\prime}$ are equivalents to the angles which any plane parallel to the two lines $S_{\circ}Z_{\circ\circ}$, $S_{\circ\circ}Z_{\circ}$, makes with the normal-chordal planes.

The interpretation of the other points, lines, angles, and planes of the figure can present no difficulty, and no further elucidation is necessary here; but in order to avoid misconceptions, it should be remembered that all through this paper (when *two* stations only are considered) we will consider the latitude of the station $S_{\circ}$ greater or not less than the latitude of the station $S_{\circ\circ}$,—as indicated in the figure.

NOTATION.

$l_{\prime}$, $l_{\prime\prime}$	denote the	latitudes of the stations $S_{\circ}$, $S_{\circ\circ}$, respectively.
l', l''	,,	colatitudes, or the arcs $PS_{\prime}$, $PS_{\prime\prime}$,,
L', L''	,,	arcs $PD_{\prime}$, $PD_{\prime\prime}$.
$A_{\prime}$, $A_{\prime\prime}$	,,	azimuths or angles $PS_{\prime}D_{\prime\prime}$, $PS_{\prime\prime}D_{\prime}$.
$A_{\circ}$, $A_{\circ\circ}$	,,	angles $PS_{\prime}S_{\prime\prime}$, $PS_{\prime\prime}S_{\prime}$ of the triangle $S_{\prime}PS_{\prime\prime}$.
$D_{\prime}$, $D_{\prime\prime}$	,,	,, $PD_{\prime}S_{\prime\prime}$, $PD_{\prime\prime}S_{\prime}$.
$z_{\prime}$, $z_{\prime\prime}$	,,	arcs $S_{\prime}D_{\prime\prime}$, $S_{\prime\prime}D_{\prime}$.

$\alpha_{,}$, $\alpha_{,,}$ denote the angles of depression of the chord, or arcs $S_{,}E_{,}$, $S_{,,}E_{,,}$.		
$\delta_{,}$, $\delta_{,,}$	„	the small arcs $S_{,}D_{,}$, $S_{,,}D_{,,}$.
$\Omega_{,}$, $\Omega_{,,}$	„	angles $S_{,,}S_{,}D_{,,}$, $S_{,}S_{,,}D_{,}$.
$R_{,}$, $R_{,,}$	„	normals $S_{o}Z_{o}$, $S_{oo}Z_{oo}$, terminating in polar axis.
$Q_{,}$, $Q_{,,}$	„	lines $S_{o}Z_{oo}$, $S_{oo}Z_{oo}$.
$\phi_{,}$, $\phi_{,,}$	„	angles $IPS_{,}$ and supplement of $IPS_{,,}$.
s, k	„	lengths of geodesic arc and chord respectively.
ν	denotes the	arc $S_{,}S_{,,}$, or the angle between the normals.
Σ	„	circular measure of the geodesic arc s.
θ	„	arc PI, or angle between the chord and polar axis.
Δ	„	angle $S_{,}IS_{,,}$ between the normal-chordal planes.
a	„	length of the earth's equatorial radius.
b	„	„ „ „ polar radius.
e	„	earth's eccentricity.

1. Values of geodetic constants, in accordance with the dimensions of the earth as finally adopted by the Ordnance Department of Great Britain and Ireland.

a = 20926348 feet	log. a = 7·3206934433
b = 20855233 feet	log. b = 7·3192150463
e = ·0823719976978	log. e = $\bar{2}$·9157795987
e^2 = ·0067851460047	log. e^2 = $\bar{3}$·8315591974
$(1-e^2)$ = ·9932148539953	log. $(1-e^2)$ = $\bar{1}$·9970432059
$\left(\frac{1}{1-e^2}\right)$ = 1·0068314987210	log. $\left(\frac{1}{1-e^2}\right)$ = 0 0029567941
$\left(\frac{e^2}{1-e^2}\right)$ = ·0068314987230	log. $\left(\frac{e^2}{1-e^2}\right)$ = $\bar{3}$·8345159915

☞ The geodetic tables above referred to give also the logs. to 8 places of decimals of the normals terminating in the polar axis for all latitudes from the equator to the pole. The well-known formula by means of which any of these normals is expressed in terms of the latitude to which it pertains is—

$$R = \frac{a}{\sqrt{1-e^2 \sin^2 l}}$$

2. The following relations are evident from the figure—

$$C_{o}p_{,} = R_{,} \cos l_{,}\,; \qquad C_{o}p_{,,} = R_{,,} \cos l_{,,} \qquad (1)$$
$$S_{o}p_{,} = R_{,} (1-e^2) \sin l_{,}\,; \qquad S_{oo}p_{,,} = R_{,,} (1-e^2) \sin l_{,,} \qquad (2)$$
$$C_{o}Z_{o} = R_{,} e^2 \sin l_{,}\,; \qquad C_{o}Z_{oo} = R_{,,} e^2 \sin l_{,,} \qquad (3)$$
$$Q^2_{,} = (C_{o}p_{,})^2 + (S_{o}p_{,} + C_{o}Z_{oo})^2 = R^2_{,} - 2R_{,} e^2 \sin^2 l_{,} + F \qquad (4)$$
$$Q^2_{,,} = (C_{o}p_{,}) + (S_{oo}p_{,,} + C_{o}Z_{o})^2 = R^2_{,,} - 2R_{,,} e^2 \sin^2 l_{,,} + F \qquad (5)$$

in which F is the same function of the latitudes in the equation (4) and (5).

$$S_{\circ}p_{,} - S_{\circ\circ}p_{,,} = (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \cdot (1-e^2) \qquad (6)$$

$$C_{\circ}Z_{\circ} - C_{\circ}Z_{\circ\circ} = (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \cdot e^2 \qquad (7)$$

$$S_{\circ}p_{,} - S_{\circ\circ}p_{,,} : Z_{\circ}Z_{\circ\circ} :: (1-e^2) : e^2 \qquad (8)$$

3. From the expressions for the magnitudes of $Q_{,}$, $Q_{,,}$, we have

$$R_{,}^2 + Q_{,}^2 = 2 \cdot R_{,}^2 (1 - e^2 \sin^2 l_{,}) + F = 2a^2 + F;$$

$$R_{,,}^2 + Q_{,,}^2 = 2 \cdot R_{,,}^2 (1 - e^2 \sin^2 l_{,,}) + F = 2a^2 + F.$$

And therefore it is obvious that we have the relation—

$$R_{,}^2 + Q_{,}^2 = R_{,,}^2 + Q_{,,}^2 \qquad (9)$$

Hence it follows that if N be the middle point of the segment $Z_{\circ}Z_{\circ\circ}$ of the polar axis intercepted by the normals, we have—

$$NS_{\circ} = NS_{\circ\circ} \qquad (10)$$

And from this it is obvious that the stations $S_{\circ}$, $S_{\circ\circ}$, are in the surface of a sphere whose centre is N, and that we have

$$R_{,} \; 7 \; Q_{,,}$$
$$R_{,,} \; 7 \; Q_{,} \qquad (11)$$
$$\delta_{,,} \; 7 \; \delta_{,}$$

(See formulæ 81·A and 81·B in the sequel.)

4. If in each of the triangles $Z_{\circ}Z_{\circ\circ}S_{\circ}$, $Z_{\circ}Z_{\circ\circ}S_{\circ\circ}$, we express the base $Z_{\circ}Z_{\circ\circ}$ in terms of the other two sides and the included angle, it is evident from (9) that—

$$R_{,} \cdot Q_{,} \cdot \cos \delta_{,} = R_{,,} \cdot Q_{,,} \cdot \cos \delta_{,,} \qquad (12)$$

$$\therefore \frac{\cos \delta_{,}}{\cos \delta_{,,}} = \frac{R_{,,} \cdot Q_{,,}}{R_{,} \cdot Q_{,}}$$

$$\therefore R_{,,} \cdot Q_{,,} \; 7 \; R_{,} \cdot Q_{,} \qquad (13)$$

absolutely; but in all ordinary cases they are equals to at least 10 places of decimals in their logarithms.

5. It is evident that the plane through the middle point N, of the segment $Z_{\circ}Z_{\circ\circ}$, perpendicular to the geodesic chord $S_{\circ}S_{\circ\circ}$, must bisect this chord or pass through its middle point M. And therefore, since the portions $NZ_{\circ}$, $NZ_{\circ\circ}$, of $Z_{\circ}Z_{\circ\circ}$, which lie on opposite sides of this plane are equals, it follows that the planes through $Z_{\circ}$, $Z_{\circ\circ}$, perpendicular to the geodesic chord $S_{\circ}S_{\circ\circ}$, cut it in points $T_{\circ}$, $T_{\circ\circ}$, equidistant from its middle point M. Hence—

$$\sin \alpha_{,} = \cos T_{\circ}S_{\circ}Z_{\circ} = \frac{S_{\circ}T_{\circ}}{R_{,}}$$

$$\sin \alpha_{,,} = \cos T_{\circ\circ}S_{\circ\circ}Z_{\circ\circ} = \frac{S_{\circ\circ}T_{\circ\circ}}{R_{,,}}$$

$$\therefore \frac{\sin \alpha_{,}}{\sin \alpha_{,,}} = \frac{R_{,,}}{R_{,}} \qquad (14)$$

And since we suppose $l_{,}$ greater than $l_{,,}$, we know that $R_{,}$ is greater than $R_{,,}$; and hence we learn that the angle of depression $\alpha_{,,}$ adjacent to the station having the lesser latitude is greater than the angle of depression $\alpha_{,}$ adjacent to the station having the greater latitude.

6. We have, evidently—

$$\frac{S_\circ T_\circ}{S_{\circ\circ} T_\circ} = \frac{S_{\circ\circ} T_{\circ\circ}}{S_\circ T_{\circ\circ}}$$

or, which is the same—

$$\frac{\tan \alpha_{,}}{\tan (z_{,} - \alpha_{,})} = \frac{\tan \alpha_{,,}}{\tan (z_{,,} - \alpha_{,,})} \qquad (15)$$

Now it is evident that each side of this equation is greater than unity; and $\therefore$ when $z_{,}$ and $z_{,,}$ are each less than a quadrant, we have—

$$\begin{aligned} \alpha_{,} &> z_{,} - \alpha_{,} \\ \alpha_{,,} &> z_{,,} - \alpha_{,,} \end{aligned} \qquad (16)$$

7. If the latitudes $l_{,}$, $l_{,,}$ of any two stations (on the same side of the earth's equator) be of constant magnitudes, then, no matter how otherwise the stations may vary in position, it is evident that the points $Z_\circ$, $Z_{\circ\circ}$, in which the normals cut the polar axis, remain fixed. It is also evident that as regards the magnitudes of L', L'', $\delta_{,}$, $\delta_{,,}$, they too are constants, and the same as if the stations were on one meridian. Hence it is obvious that when $l_{,}$ is greater than $l_{,,}$, or, which is the same—when l'' is greater than l', we know that the first and third of the following are true—

$$\begin{aligned} l'' &> L'' \\ L'' &> L' \\ L' &> l' \end{aligned} \qquad (17)$$

The truth of the second of these relations is easily seen. For drawing perpendiculars $S_\circ H_\circ$, $S_{\circ\circ} H_{\circ\circ}$, from the stations to the polar axis, it is evident we have—

$$\begin{aligned} \tan L'' &= S_{\circ\circ} H_{\circ\circ} \div (Z_{\circ\circ} H_{\circ\circ} + Z_{\circ\circ} Z_\circ) \\ \tan L' &= S_\circ H_\circ \div (Z_{\circ\circ} H_{\circ\circ} + H_{\circ\circ} H_\circ); \end{aligned}$$

and therefore since $S_{\circ\circ} H_{\circ\circ} > S_\circ H_\circ$, and that $Z_{\circ\circ} Z_\circ < H_{\circ\circ} H_\circ$,

$$\begin{aligned} \tan L'' &> \tan L' \\ L'' &> L' \end{aligned}$$

Hence also (since each of the four arcs is less than 90°) we have

$$\begin{aligned} \sin l'' &> \sin L'' \\ \sin L'' &> \sin L' \\ \sin L' &> \sin l' \end{aligned} \qquad (18)$$

8. From the spherical triangles $D_{\prime}PS_{\prime\prime}$, $D_{\prime\prime}PS_{\prime}$ we have—

$$\begin{aligned} \sin L' \sin D_{\prime} &= \sin l'' \sin A_{\prime\prime} \\ \sin L'' \sin D_{\prime\prime} &= \sin l' \sin A_{\prime} \end{aligned}$$

∴

$$\begin{aligned} \sin D_{\prime} &> \sin A_{\prime\prime} \\ \sin A_{\prime} &> \sin D_{\prime\prime} \end{aligned} \qquad (19)$$

And since each of the angles $(D_{\prime} + A_{\prime\prime})$, $(A_{\prime} + D_{\prime\prime})$, is less than 180°, it follows that—

$$\begin{aligned} &D_{\prime} > A_{\prime\prime}, \text{ and that } A_{\prime\prime} \text{ is acute} \\ &A_{\prime} > D_{\prime\prime}, \text{ and that } D_{\prime\prime} \text{ is acute} \end{aligned} \qquad (20)$$

9. We shall now establish the following important relations between the azimuths and angles $D_{\prime}$, $D_{\prime\prime}$—

$$\begin{aligned} D_{\prime} &> A_{\prime} \\ A_{\prime} &> A_{\prime\prime} \\ A_{\prime\prime} &> D_{\prime\prime} \end{aligned} \qquad (21)$$

First, from the triangles $S_{\prime}PD_{\prime\prime}$, $S_{\prime\prime}PD_{\prime}$, we have—

$$\begin{aligned} \sin z_{\prime} \sin A_{\prime} &= \sin L'' \sin \omega \\ \sin z_{\prime\prime} \sin A_{\prime\prime} &= \sin L' \sin \omega \end{aligned}$$

But from (14), (15), and (16), it is evident that—

$$z_{\prime\prime} > z_{\prime} \qquad (22)$$

And therefore, since $\sin L''$ is greater than $\sin L'$ we have—

$$\sin z_{\prime} \sin A_{\prime} > \sin z_{\prime\prime} \sin A_{\prime\prime}$$

$$\therefore \quad \frac{\sin A_{\prime}}{\sin A_{\prime\prime}} > 1.$$

Now, since $A_{\prime} + A_{\prime\prime}$ is less than 180°, and that angle $A_{\prime\prime}$ is acute (see 20), therefore it follows that—

$$A_{\prime} > A_{\prime\prime}$$

In order to shew that the first and third of the relations (21) are true, we may proceed thus—

Applying formula 4, page 158, of Serret's Trigonometry to the spherical triangle $S_{\prime}IS_{\prime\prime}$, and putting ϵ to represent the spherical excess of this triangle, we have—

$$\tan \tfrac{1}{2} (\triangle - \epsilon) = \frac{\sin \frac{1}{2} (\alpha_{\prime} - \alpha_{\prime\prime})}{\cos \frac{1}{2} (\alpha_{\prime} + \alpha_{\prime\prime})} \cdot \tan \tfrac{1}{2} \triangle \qquad (23)$$

And, since $a_{,}—a_{,,}$ is negative, it follows $\triangle$ is less than ϵ; Hence also—

$$\text{angle } IS_{,}S_{,,} + \text{angle } IS_{,,}S_{,} \ 7 \ 180°$$
$$\text{angle } S_{,}S_{,,}D_{,} \ 7 \ \text{angle } S_{,,}S_{,}D_{,,}$$

or, $$\Omega_{,,} \ 7 \ \Omega_{,} \tag{24}$$

We have also—

$$A_{,} + A_{,,} = PS_{,}S_{,,} + PS_{,,}S_{,} + (\epsilon - \triangle)$$

& ∴ $$A_{,} + A_{,,} \ 7 \ A_{\circ} + A_{\circ\circ} \tag{25}$$

Now the triangle $S_{,}ID_{,}$ is evidently such that—

$$\text{angle } IS_{,}D_{,} + \text{angle } ID_{,}S_{,} \angle 180°$$

but, $$\text{angle } PD_{,}S_{,,} + \text{angle } ID_{,}S_{,} = 180°$$

∴, $$\text{angle } PD_{,}S_{,,} \ 7 \ \text{angle } IS_{,}D_{,}$$

or, $$D_{,} \ 7 \ A_{,}$$

And the triangle $S_{,,}ID_{,,}$ is evidently such that—

$$\text{angle } IS_{,,}D_{,,} + \text{angle } ID_{,}S_{,,} \ 7 \ 180$$

but, $$\text{angle } PD_{,,}S_{,} + \text{angle } ID_{,,}S_{,,} = 180$$

∴, $$\text{angle } IS_{,,}D_{,,} \ 7 \ \text{angle } PD_{,,}S_{,}$$

or, $$A_{,,} \ 7 \ D_{,,}$$

10. From equation (14) or, $\frac{\sin a_{,}}{\sin a_{,,}} = \frac{R_{,,}}{R_{,}}$, we have—

$$\frac{\sin a_{,,} - \sin a_{,}}{\sin a_{,,} + \sin a_{,}} = \frac{R_{,} - R_{,,}}{R_{,} + R_{,,}}$$

$$\frac{\tan \frac{1}{2}(a_{,,} - a_{,})}{\tan \frac{1}{2}(a_{,,} + a_{,})} = \frac{R_{,} - R_{,,}}{R_{,} + R_{,,}} \tag{26}$$

$$\tan \tfrac{1}{2}(a_{,,} - a_{,}) = \frac{R_{,} - R_{,,}}{R_{,} + R_{,,}} \tan \tfrac{1}{2} \Sigma \tag{27}$$

From this equation it is evident that when the latitudes are of constant magnitudes, then the greater the circular measure Σ of the intervening geodesic arc is, the greater will be the difference of the angles of depression of the chord. But although $a_{,,}—a_{,}$ increases or decreases according as Σ increases or decreases, it is nevertheless evident, from (14), that both $a_{,,}$ and $a_{,}$ increase or decrease as $a_{,,} + a_{,}$ or Σ increases or decreases.

Moreover, it is evident that when the latitudes are constants—

$$\frac{\cos a_{,}}{\cos a_{,,}} \text{ increases as } \Sigma \text{ increases} \tag{28}$$

$$\frac{\tan a_{,}}{\tan a_{,,}} \text{ decreases as } \Sigma \text{ increases} \tag{29}$$

However, it is proper to observe that even for a geodesic arc on the earth's spheroidal surface whose circular measure is as great as $1°_{\prime\prime}\ 30'$, and the latitudes of whose extremities differ by as much as 1°, we may, with due respect to the utmost attainable precision in geodetic surveying in Victoria, assume—

$$\frac{\cos \alpha_{\prime}}{\cos \alpha_{\prime\prime}} = 1 \qquad (30)$$

For by means of (27) it can be easily shown that even in this extreme case $\alpha_{\prime\prime} - \alpha_{\prime}$ is less than a sixth part of a second, and that the logarithms of $\cos \alpha_{\prime}$ and $\cos \alpha_{\prime\prime}$ will be the same to 8 places of decimals, and differ in the ninth place by less than 4. Hence also, in the actual practice of trigonometrical surveying, we may, for some purposes, assume—

$$\frac{\alpha_{\prime\prime}}{\alpha_{\prime}} = \frac{\tan \alpha_{\prime\prime}}{\tan \alpha_{\prime}} = \frac{\sin \alpha_{\prime\prime}}{\sin \alpha_{\prime}} = \frac{R_{\prime}}{R_{\prime\prime}} \qquad (31)$$

$$\alpha_{\prime\prime} - \alpha_{\prime} = \frac{R_{\prime} - R_{\prime\prime}}{R_{\prime} + R_{\prime\prime}} \cdot \Sigma \qquad (32)$$

their logs. being the same to at least 8 places of decimals. Formulæ 27 and 32 will be found very useful in the computation of the angles of depression of the chord of the geodesic arc; but, when worked by means of logarithms, the best way is to find, in the first instance, an angle x such that—

$$\tan x = \frac{R_{\prime}}{R_{\prime\prime}} \qquad (33)$$

and then equations (27) and (32) can be written in the forms—

$$\tan \tfrac{1}{2} (\alpha_{\prime\prime} - \alpha_{\prime}) = \tan (x - 45°) \cdot \tan \tfrac{1}{2} \Sigma \qquad (34)$$

$$\alpha_{\prime\prime} - \alpha_{\prime} = \tan (x - 45°) \cdot \Sigma'' \qquad (35)$$

And since the angle $x - 45°$ can never be more than a few seconds in magnitude we have, in lieu of 35—

$$\alpha_{\prime\prime} - \alpha_{\prime} = \Sigma'' \cdot (x - 45°) \sin 1'' \qquad (36)$$

Moreover, it is evident, that in actual practice, we infer—from (31) and (15)—that—

$$\frac{\alpha_{\prime}}{z_{\prime} - \alpha_{\prime}} = \frac{\alpha_{\prime\prime}}{z_{\prime\prime} - \alpha_{\prime\prime}} \text{ approximately} \qquad (37)$$

and ∴

$$\frac{z_{\prime}}{z_{\prime\prime}} = \frac{\alpha_{\prime}}{\alpha_{\prime\prime}} = \frac{\sin \alpha_{\prime}}{\sin \alpha_{\prime\prime}} = \frac{R_{\prime\prime}}{R_{\prime}} \qquad (38)$$

shewing that the auxiliary angle x of (33) has its tangent equal to the ratio of the angles of depression of the chord, and also equal to the ratio of the arcs $z_{\prime\prime}$ and $z_{\prime}$.

11. Again, from the triangle $S_,IS_{,,}$, we have, rigorously—

$$\frac{\sin \Omega_,}{\sin \Omega_{,,}} = \frac{\cos \alpha_{,,}}{\cos \alpha_,} \qquad (39)$$

Hence it follows that for any pair of mutually visible stations, such as occur in trigonometrical surveying, we may assume—

$$\left.\begin{array}{l} \dfrac{\sin \Omega_,}{\sin \Omega_{,,}} = 1\,; \\ \dfrac{\tan \Omega_,}{\tan \Omega_{,,}} = 1\,; \\ \dfrac{\cos \Omega_,}{\cos \Omega_{,,}} = 1\,; \end{array}\right\} \text{their logarithms being the same to at least 8 places of decimals.} \qquad (40)$$

(See formulæ (30) and remarks as to its approximate accuracy.)

12. From what has been already shewn or observed, it is evident—

$$\Omega_{,,} - \Omega_, = \epsilon - \triangle \qquad (41)$$

and ∴, we have from (23)—

$$\tan \tfrac{1}{2}(\Omega_{,,} - \Omega_,) = \frac{\sin \frac{1}{2}(\alpha_{,,} - \alpha_,)}{\cos \frac{1}{2}\Sigma} \cdot \tan \tfrac{1}{2}\triangle \qquad (42)$$

∴

$$\Omega_{,,} - \Omega_, = \frac{\sin \frac{1}{2}(\alpha_{,,} - \alpha_,)}{\cos \frac{1}{2}\Sigma} \cdot \triangle \qquad (43)$$

and, since $\alpha_{,,} - \alpha_,$ is but a fraction of a second, even when Σ is as much as $1^\circ_{,,}\ 30'$; and that $\triangle$ can be but a few seconds in all cases that occur; it is easy to prove that, in the actual practice of trigonometrical surveying, the angle $\Omega_{,,} - \Omega_,$ will never exceed the $\frac{1}{10000}$ part of a second. And from this and equations (40) it follows that we can regard

$$\Omega_{,,} = \Omega_, = \Omega$$

In the account of the trigonometrical survey of Great Britain and Ireland, the magnitude of $\Omega_{,,} - \Omega_,$ is shewn to be always less than $\frac{1}{10000}$ part of a second; but it is not shewn that the ratio of the sines or tangents of the angles $\Omega_{,,}$, $\Omega_,$, may be regarded as equal to unity for all pairs of mutually visible stations: yet this is necessary, as, in some instances, $\Omega_{,,}$ and $\Omega_,$ are extremely small arcs.

13. And if we put $\Xi_,$ and $\Xi_,$ to represent the small spherical angles $S_{,,}D_,D_{,,,}$, $S_,D_{,,}D_{,,}$ it is evident that, in like manner, we have—

$$\Xi_{,,} - \Xi_, = \frac{\sin \frac{1}{2}(D_,E_{,,} - D_{,,}E_,)}{\cos \frac{1}{2}(D_,E_{,,} + D_{,,}E_,)} \cdot \triangle \qquad (44)$$

and it can be easily shewn that the difference of the angles $\Xi_{,,}$ and $\Xi_,$ is as extremely small as the difference of the angles $\Omega_{,,}$ and $\Omega_{,}$, and that they too can be regarded as equal to each other. Moreover, the points $D_{,}OD_{,,}$ are on one great circle.

14. Now, since for all pairs of mutually visible stations on the earth's spheroidal surface, we have—

$$A_{,} + A_{,,} = A_{\circ} + A_{\circ\circ}$$

and that we can express the angle ω in terms of the angles $A_{\circ} + A_{\circ\circ}$ and the sides l', l'', of the triangle $S_{,}PS_{,,}$; therefore by substituting, in such expression, $A_{,} + A_{,,}$ for its equivalent, we have—

$$\tan \tfrac{1}{2}\,\omega = \frac{\cos \frac{1}{2}\,(l'' - l')}{\cos \frac{1}{2}\,(l'' + l')} \cdot \cot \tfrac{1}{2}\,(A_{,} + A_{,,})$$

$$\therefore \qquad (45)$$

$$\tan \tfrac{1}{2}\,\omega = \frac{\cos \frac{1}{2}\,(l_{,} - l_{,,})}{\sin \frac{1}{2}\,(l_{,} + l_{,,})} \cdot \cot \tfrac{1}{2}\,(A_{,} + A_{,,})$$

This formulæ is known as *Dalby's Theorem*, for the history of which see the "Account of the Principal Triangulation of Great Britain and Ireland," page 236.

15. By applying Delambre's analogies to the same spherical triangle $S_{,}PS_{,,}$, we find in like manner—

$$\sin \tfrac{1}{2}\,(A_{,} + A_{,,}) = \frac{\cos \frac{1}{2}\,\omega}{\cos \frac{1}{2}\,\nu} \cdot \cos \tfrac{1}{2}\,(l'' - l') \qquad (46)$$

$$\cos \tfrac{1}{2}\,(A_{,} + A_{,,}) = \frac{\sin \frac{1}{2}\,\omega}{\cos \frac{1}{2}\,\nu} \cdot \cos \tfrac{1}{2}\,(l'' + l') \qquad (47)$$

and $\therefore$

$$\tan \tfrac{1}{2}\,(A_{,} + A_{,,}) = \frac{\cos \frac{1}{2}\,(l'' - l')}{\cos \frac{1}{2}\,(l'' + l')} \cdot \cot \tfrac{1}{2}\,\omega$$

$$(48)$$

$$\cot \tfrac{1}{2}\,(A_{,} + A_{,,}) = \frac{\cos \frac{1}{2}\,(l'' + l')}{\cos \frac{1}{2}\,(l'' - l')} \cdot \tan \tfrac{1}{2}\,\omega$$

☞ From (48) it is evident that when the latitudes of the stations are of constant magnitudes, then the *greater* the difference of longitude ω is, the *less* will the sum of the two azimuths be.

"CONVERGENCE OF MERIDIANS."

The stations being supposed on the same side of the earth's equator, the sum of the azimuths $A_{,} + A_{,,}$ is always less than 180°; and it is customary to call the defect or

$$180° - (A_{,} + A_{,,})$$

the "convergence" of the meridians as respects the stations. Putting C to denote this convergence, it is evident from 48 that we have—

$$\tan \tfrac{1}{2}\,C = \frac{\sin \frac{1}{2}\,(l_{,} + l_{,,})}{\cos \frac{1}{2}\,(l_{,} - l_{,,})} \cdot \tan \tfrac{1}{2}\,\omega$$

And should the latitudes of the stations be equal, then putting l for the common value, we have the rigorous formula

$$\tan \tfrac{1}{2} C = \sin l \cdot \tan \tfrac{1}{2} \omega$$

or, since the tangents of small angles are proportional to the numbers of seconds in the angles, we have, approximately—

$$C'' = \sin l \cdot \omega''$$

in which C'' and ω'' represent the seconds in the "convergence" of meridians, and in the difference of the longitude of the stations.

16. And applying Todhunter's formula pertaining to spherical excess (see page 72, formula 3, of his trigonometry) to the same spherical triangle, we at once obtain the useful relations—

$$\left.\begin{aligned} \cot \tfrac{1}{2} l' \cdot \cot \tfrac{1}{2} l'' &= -\frac{\cos \tfrac{1}{2} (A_{,} + A_{,,} - \omega)}{\cos \tfrac{1}{2} (A_{,} + A_{,,} + \omega)} \\ \tan \tfrac{1}{2} l' \cdot \tan \tfrac{1}{2} l'' &= -\frac{\cos \tfrac{1}{2} (A_{,} + A_{,,} + \omega)}{\cos \tfrac{1}{2} (A_{,} + A_{,,} - \omega)} \end{aligned}\right\} \quad (49)$$

It is evident that instead of $\tfrac{1}{2} l'$ and $\tfrac{1}{2} l''$, we may write $(45° - \tfrac{1}{2} l_{,})$ and $(45° - \tfrac{1}{2} l_{,,})$ in formulæ (49).

17. From the spherical triangles S,PI, S,,PI, we have—

$$\sin \phi_{,} = \frac{\sin A_{,} \cos \alpha_{,}}{\sin \theta}; \qquad \sin \phi_{,,} = \frac{\sin A_{,,} \cos \alpha_{,,}}{\sin \theta}$$

∴

$$\frac{\sin A_{,}}{\sin A_{,,}} = \frac{\sin \phi_{,}}{\sin \phi_{,,}} \cdot \frac{\cos \alpha_{,,}}{\cos \alpha_{,}}$$

But from the plane triangle $p_{,}C_{\circ}p_{,,}$, we have—

$$\frac{\sin \phi_{,}}{\sin \phi_{,,}} = \frac{R_{,,} \cos l_{,,}}{R_{,} \cos l_{,}}$$

∴ also the rigorous formula—

$$\frac{\sin A_{,}}{\sin A_{,,}} = \frac{R_{,,} \cos l_{,,}}{R_{,} \cos l_{,}} \cdot \frac{\cos \alpha_{,,}}{\cos \alpha_{,}} \quad (50)$$

And since for any pair of mutually visible stations, such as occur in trigonometrical surveying, we may assume $\frac{\cos \alpha_{,,}}{\cos \alpha_{,}} = 1$,

∴ we have—

$$\frac{\sin A_{,}}{\sin A_{,,}} = \frac{R_{,,} \cos l_{,,}}{R_{,} \cos l_{,}} \quad (51)$$

$$\frac{\sin A_{,}}{\sin A_{,,}} = \frac{\cos l_{,,}}{\cos l_{,}} \sqrt{\frac{1 - e^2 \sin^2 l_{,}}{1 - e^2 \sin^2 l_{,,}}} \quad (52)$$

$$\frac{\sin^2 A_{,}}{\sin^2 A_{,,}} = \frac{(1 - e^2) \tan^2 l_{,} + 1}{(1 - e^2) \tan^2 l_{,,} + 1} \quad (53)$$

(true to at least 8 decimals places in their logs.)

☞ From either of these we at once perceive that, with respect to mutually visible stations, *the ratio of the sines of the azimuths* will remain sensibly constant when the latitudes of the stations are of constant magnitudes, no matter how the difference of longitude or the intervening geodesic arc may vary in magnitude.

18. If we find an angle σ such that—

$$\tan \sigma = \frac{R_{\prime\prime} \cos l_{\prime\prime}}{R_{\prime} \cos l_{\prime}} \qquad (54)$$

then from 51, we derive—

$$\frac{\tan \frac{1}{2} (A_{\prime} - A_{\prime\prime})}{\tan \frac{1}{2} (A_{\prime} + A_{\prime\prime})} = \tan (\sigma - 45^\circ) \qquad (55)$$

$$\therefore \tan \tfrac{1}{2} (A_{\prime} - A_{\prime\prime}) = \tan \tfrac{1}{2} (A_{\circ} + A_{\circ\circ}) \cdot \tan (\sigma - 45^\circ) \qquad (56)$$

$$\tan \tfrac{1}{2} (A_{\prime} - A_{\prime\prime}) = \frac{\cos \frac{1}{2} (l_{\prime} - l_{\prime\prime})}{\sin \frac{1}{2} (l_{\prime} + l_{\prime\prime})} \cdot \tan (\sigma - 45^\circ) \cdot \cot \tfrac{1}{2} \omega \qquad (57)$$

☞ From this equation it is evident that when the latitudes are constants, then the greater ω is, the *less* will the difference of the azimuths be. We already know that, in such case, the *less* also will be the sum of the azimuths, and $\therefore$ the less will each of the azimuths be.

19. It is evident that $A_{\circ} - A_{\circ\circ} = A_{\prime} - A_{\prime\prime} + 2\,\Omega$ and $\therefore$

$$\tan \left\{ \tfrac{1}{2} (A_{\prime} - A_{\prime\prime}) + \Omega \right\} = \frac{\sin \frac{1}{2} (l_{\prime} - l_{\prime\prime})}{\cos \frac{1}{2} (l_{\prime} + l_{\prime\prime})} \cdot \cot \tfrac{1}{2} \omega \qquad (58)$$

and from this and (57) it is evident that when the latitudes of the stations are constants in magnitude, we have

$$\frac{\tan \left\{ \frac{1}{2} (A_{\prime} - A_{\prime\prime}) + \Omega \right\}}{\tan \frac{1}{2} (A_{\prime} - A_{\prime\prime})} = \text{constant.}$$

And since the greater the difference of longitude of the stations is, the less $A_{\prime} - A_{\prime\prime}$ must be; $\therefore$ the greater ω is, the less will Ω be.

20. From the spherical triangle $S_{\prime}PS_{\prime\prime}$, we have

$$\frac{\sin (A_{\prime\prime} - \Omega)}{\sin (A_{\prime} + \Omega)} = \frac{\sin l'}{\sin l''}$$

$$\therefore \tan \Omega = \frac{\sin A_{\prime\prime} \sin l'' - \sin A_{\prime} \sin l'}{\cos A_{\prime\prime} \sin l'' + \cos A_{\prime} \sin l'} \qquad (59)$$

☞ In such cases as occur in trigonometrical surveying the angle Ω will range from zero to a limiting value of about $10'_{\prime\prime}\ 00''$. In the case of the worked-out example in the sequel, the value of Ω is $7'_{\prime\prime}\ 22''$ nearly.

21. From the spherical triangles $S_{\prime}PI$, $S_{\prime\prime}PI$, we have—

$$\sin \theta \sin \phi_{\prime} = \sin A_{\prime} \cos \alpha_{\prime}$$
$$\sin \theta \sin \phi_{\prime\prime} = \sin A_{\prime\prime} \cos \alpha_{\prime\prime}$$

Multiplying both sides of these equations by the chord k, and remembering that the projection $k_\circ$ of the chord on the plane of the equator is equal to $k.$ sin θ, we have—

$$k \cdot \sin A_{,} \cos \alpha_{,} = k_\circ \cdot \sin \phi_{,}$$
$$k \cdot \sin A_{,,} \cos \alpha_{,,} = k_\circ \cdot \sin \phi_{,,}$$

But from the plane triangle $p_{,}C_\circ p_{,,}$ we know that

$$k_\circ = \frac{R_{,} \cos l_{,} \sin \omega}{\sin \phi_{,}} = \frac{R_{,,} \cos l_{,,} \sin \omega}{\sin \phi_{,,}}$$

∴ we have—

$$\left.\begin{array}{l} k \cdot \sin A_{,} \cos \alpha_{,} = R_{,,} \cos l_{,,} \sin \omega \\ k \,.\, \sin A_{,,} \cos \alpha_{,,} = R_{,} \cos l_{,} \sin \omega \end{array}\right\} \quad (60)$$

And, since $k = 2s \cdot \sin \frac{1}{2} \Sigma \div \Sigma \cdot \sin 1''$, we have—

$$\frac{2s \cdot \sin A_{,} \sin \frac{1}{2} \Sigma \cdot \cos \alpha_{,}}{\Sigma \cdot \sin 1''} = R_{,,} \cos l_{,,} \sin \omega \quad (61)$$

$$\frac{2s \cdot \sin A_{,,} \sin \frac{1}{2} \Sigma \cdot \cos \alpha_{,,}}{\Sigma \cdot \sin 1''} = R_{,} \cos l_{,} \sin \omega$$

And since for any pair of mutually visible stations $\cos \alpha_{,} = \cos \alpha_{,,} = \cos \frac{1}{2} \Sigma$,

$$\left.\begin{array}{l} \dfrac{s \cdot \sin A_{,} \cdot \sin \Sigma}{\Sigma \cdot \sin 1''} = R_{,,} \cos l_{,,} \sin \omega \\ \dfrac{s \cdot \sin A_{,,} \sin \Sigma}{\Sigma \cdot \sin 1''} = R_{,} \cos l_{,} \sin \omega \end{array}\right\} \quad (62)$$

When the geodesic arc s is such that its circular measure Σ is not more than 1°, we immediately deduce the relations—

$$\left.\begin{array}{l} \omega = \dfrac{s \cdot \sin A_{,}}{R_{,,} \cdot \cos l_{,,} \cdot \sin 1''} \\ \omega = \dfrac{s \cdot \sin A_{,,}}{R_{,} \cdot \cos l_{,} \cdot \sin 1''} \end{array}\right\} \quad (63)$$

☞ In Chambers' "Practical Mathematics," and in the article on "Geodesy" in Spon's Dictionary of Engineering, the formulæ (63) are given in an erroneous form which must inevitably lead to incompatible results when applied in trigonometrical surveying. The erroneous formulæ given there and elsewhere are—

$$\omega = \frac{s \cdot \sin A_{,}}{R_{,} \cdot \cos l_{,,} \cdot \sin 1''} = \frac{s \cdot \sin A_{,,}}{R_{,,} \cdot \cos l_{,} \cdot \sin 1''}$$

(See note 6 to problem 10 given in the sequel.)

22. From 50 or 60 we have—

$$\frac{\cos \alpha_{,}}{\cos \alpha_{,,}} = \frac{R_{,,} \cos l_{,,} \sin A_{,,}}{R_{,} \cos l_{,} \sin A_{,}} \quad (64)$$

But (14)
$$\frac{\sin \alpha_{,}}{\sin \alpha_{,,}} = \frac{R_{,,}}{R_{,}} \qquad (65)$$
$$\therefore \qquad \frac{\tan \alpha_{,}}{\tan \alpha_{,,}} = \frac{\cos l_{,} \sin A_{,}}{\cos l_{,,} \sin A_{,,}} \qquad (66)$$

From these we can easily express the squares of the sines, cosines, and tangents of the angles of depression of the chord in terms of the two latitudes and two azimuths; but it is obvious that such expressions must assume the indefinite form $\frac{0}{0}$ when the latitudes are equal, or $R_{,} = R_{,,}$. And from (64) and (27), we have—

$$\tan \tfrac{1}{2} (\alpha_{,,} + \alpha_{,}) = \left(\frac{R_{,} + R_{,,}}{R_{,} - R_{,,}}\right)^{\frac{1}{2}} \cdot \left(\frac{R_{,,} \cos l_{,,} \sin A_{,,} - R_{,} \cos l_{,} \sin A_{,}}{R_{,,} \cos l_{,,} \sin A_{,,} + R_{,} \cos l_{,} \sin A_{,}}\right)^{\frac{1}{2}}$$

$$\tan \tfrac{1}{2} (\alpha_{,,} - \alpha_{,}) = \left(\frac{R_{,} - R_{,,}}{R_{,} + R_{,,}}\right)^{\frac{1}{2}} \cdot \left(\frac{R_{,,} \cos l_{,,} \sin A_{,,} - R_{,} \cos l_{,} \sin A_{,}}{R_{,,} \cos l_{,,} \sin A_{,,} + R_{,} \cos l_{,} \sin A_{,}}\right)^{\frac{1}{2}}$$

The expression for $\tan \frac{1}{2} \Sigma$ or $\tan \frac{1}{2} (\alpha_{,,} + \alpha_{,})$, given in (67), is of a like character. It assumes the indefinite form $\frac{0}{0}$ when $R_{,} = R_{,,}$; which is the case on a spheroid when the latitudes of the stations are equal, and always the case on a sphere, no matter how the stations may be situated with respect to each other.

23. From the triangles $D_{,}S_{,}I$, $D_{,,}S_{,,}I$, we have—

$$\frac{\cos \alpha_{,}}{\cos (z_{,,} - \alpha_{,,})} = \frac{\sin D_{,}}{\sin A_{,}}$$
$$\frac{\cos \alpha_{,,}}{\cos (z_{,} - \alpha_{,})} = \frac{\sin D_{,,}}{\sin A_{,,}} \qquad (69)$$

$$\sin D_{,} = \frac{\cos l_{,,} \sin \omega}{\sin z_{,}}$$
$$\sin D_{,,} = \frac{\cos l_{,} \sin \omega}{\sin z_{,,}} \qquad (70)$$

And from these we at once obtain the relations—

$$\cot z_{,} = \frac{\sin A_{,,} \cos \alpha_{,,}}{\cos l_{,} \sin \omega \cos \alpha_{,}} - \tan \alpha_{,}$$
$$\cot z_{,,} = \frac{\sin A_{,} \cos \alpha_{,}}{\cos l_{,} \sin \omega \cos \alpha_{,,}} - \tan \alpha_{,,} \qquad (71)$$

If in these we substitute the values of $\sin \omega$ from (60) we have—

$$\tan z_{,} = \frac{k \cdot \cos \alpha_{,}}{R_{,} - k \sin \alpha_{,}}$$
$$\tan z_{,,} = \frac{k \cdot \cos \alpha_{,,}}{R_{,,} - k \cdot \sin \alpha_{,,}} \qquad (72)$$

From the triangles $S_oS_{oo}Z_o$, $S_{oo}S_oZ_{oo}$, we have—

$$\sin z_{,} = \frac{k \cdot \cos (z_{,} - a_{,})}{R_{,}}$$
$$\sin z_{,,} = \frac{k \cdot \cos (z_{,,} - a_{,,})}{R_{,,}} \qquad (73)$$

And for stations which do not differ in latitude by more than 1°, we know that $\cos (z_{,} - a_{,})$, $\cos (z_{,,} - a_{,,})$, and $\cos \frac{1}{2} \Sigma$, are the same to 8 places of decimals in their logarithms; ∴ for such stations we have the closely approximate formulæ—

$$\sin z_{,} = \frac{k \cdot \cos \frac{1}{2} \Sigma}{R_{,}}$$
$$\sin z'' = \frac{k \cdot \cos \frac{1}{2} \Sigma}{R''} \qquad (74)$$

But in order to find $z_{,}$ and $z_{,,}$ in the actual practice of trigonometrical surveying (the latitudes of the two stations being such as do not differ by more than 1°) we have the well-known simple formulæ—

$$z_{,} = \frac{s}{R_{,} \cdot \sin 1''}$$
$$z_{,,} = \frac{s}{R_{,,} \cdot \sin 1''} \qquad (75)$$

which enable us to find $z_{,}$ and $z_{,,}$ to within $\frac{1}{1000}$ part of a second of rigorous accuracy. This can be easily seen from the following—

We have the rigorously true equation—

$$R_{,} \cdot Q_{,} \cdot \cos \delta_{,} = R_{,,} \cdot Q_{,,} \cdot \cos \delta_{,,}$$

in which (as is shewn in the sequel) $\delta_{,}$ and $\delta_{,,}$ are always each less than 16 seconds, and differ from each other by less than 0·2″; and as we know that under such circumstances the logs. of $\cos \delta_{,}$ and $\cos \delta_{,,}$ will be the same to 10 places of decimals, ∴ we can assume—

$$R_{,} \cdot Q_{,} = R_{,,} \cdot Q_{,,}$$

But $\quad R_{,}^2 + Q_{,}^2 = R_{,,}^2 + Q_{,,}^2$ absolutely,

$$\therefore \quad R_{,} = Q_{,,} \quad \text{nearly}$$
$$R_{,,} = Q_{,} \text{ nearly}$$

Hence if $I_{,}$, $I_{,,}$ be put to represent the bases of the isosceles triangles having the angles $z_{,}$, $z_{,,}$ as vertical angles, and sides equal to $R_{,}$, $R_{,,}$ respectively, we have—

$$I_{,}^2 = R_{,}^2 + R_{,}^2 - 2\,R_{,}^2 \cos z_{,}$$
$$= R_{,}^2 + Q_{,,}^2 - 2\,R_{,} \cdot Q_{,,} \cos z_{,}$$
$$= k^2$$

and ∴, obviously, we have $z_{,} = \dfrac{s}{R_{,} \cdot \sin 1''}$

And, $$I_{,,}^2 = R_{,,}^2 + R_{,,}^2 - 2\, R_{,,}^2 \cdot \cos z_{,,}$$
$$= R_{,,}^2 + Q_{,}^2 - 2\, R_{,,} \cdot Q_{,} \cdot \cos z_{,,}$$
$$= k^2$$

∴, obviously, we have $z_{,,} = \dfrac{s}{R_{,,} \sin 1''}$

Nevertheless it is evident that the perpendicular let fall from the station S_{o} on the line $S_{oo}Z_{o}$, lies inside the triangle $S_{o}Z_{o}S_{oo}$, and that the perpendicular let fall from S_{oo} on the line $S_{o}Z_{oo}$ lies inside the triangle $S_{o}Z_{oo}S_{oo}$; and ∴ that $I_{,}\ 7\ k$, and also $I_{,,}\ 7\ k$; and that, with respect to absolute accuracy, we have—

$$z_{,}\ 7\ \frac{s}{R_{,} \sin 1''}; \qquad z_{,,}\ 7\ \frac{s}{R_{,,} \sin 1''}.$$

However, the values of $z_{,}$ and $z_{,,}$ as given by (75) are such that for a distance of a degree along the meridian they cannot differ from the absolutely true values by as much as $\frac{6}{10}$ of an inch of error in the length of s would cause. (See "Account of," &c., page 247.)

☞ It is no easy matter to guard against inferring that $z_{,,}$ can never be greater than $\dfrac{s}{\rho \cdot \sin 1''}$ or $(a_{,,} + a_{,})$. But that $z_{,,}$ can be greater than $a_{,,} + a_{,}$ may be easily seen in the following manner:—

It has been already shewn that in all cases in which $l_{,}$ is greater than $l_{,,}$, we must have $D_{,}$ greater than $A_{,}$. Now if we suppose the point S_{o} fixed on the spheroidal earth (and ∴ $S_{,}$ also fixed on the unit sphere), and that the point S_{oo} (which has $S_{,,}$ as corresponding point on the unit sphere) assumes at first a position such that $l_{,} = l_{,,}$, and then moves continuously along the meridian in which it is situated, making $l_{,,}$ less and less until the angle $A_{,}$ becomes $= 90°$, then of course $D_{,}$ from being equal to $A_{,}$ at the commencement must have increased continuously until at length it exceeded 90°. And it is evident that at one state of the implicated entities, the angle $D_{,}$ was 90°, and $A_{,}$ less than 90°, and ∴ that in such state $\sin A_{,}$ was less than $\sin D_{,}$. But if we were to assume that $z_{,,}$ should be always less than $a_{,,} + a_{,}$, or never greater than $a_{,,} + a_{,}$, then $ID_{,}$ should be always greater than $IS_{,}$, and ∴ $\sin A_{,}$ always greater than $\sin D_{,}$, which we know to be absurd.

☞ Moreover, it is evident that by putting V to represent the particular value of the angle A, when unequal to $D_{\prime}$ but such that $\sin A_{\prime} = \sin D_{\prime}$ (in which case $A_{\prime}$ is acute and $D_{\prime}$ obtuse) it is evident that—

$$\text{whenever } A_{\prime} \; 7 \; V\text{, then will } z_{\prime\prime} \angle a_{\prime\prime} + a_{\prime} \text{ or } \Sigma$$
$$\text{whenever } A_{\prime} \angle V\text{, then will } z_{\prime\prime} \; 7 \; a_{\prime\prime} + a_{\prime} \text{ or } \Sigma$$

Hence:—If $S_{\circ\circ}$ be any fixed point within any convex closed curve on the earth's spheroidal surface, and $Z_{\circ\circ}$ the point in which the normal to the surface at $S_{\circ\circ}$ cuts the polar axis: then there are 4 real points $S_{\circ}$ on this curve, and 4 only, such that the angle $S_{\circ\circ}Z_{\circ\circ}S_{\circ}$ subtended at $Z_{\circ\circ}$ is equal to the sum of the angles $a_{\prime\prime}$, $a_{\prime}$, of depression of the chord $S_{\circ\circ}S_{\circ}$ below the tangent planes at $S_{\circ\circ}$, $S_{\circ}$. Viz.—The two points in which the curve is cut by the plane X through $S_{\circ\circ}$ which is perpendicular to the polar axis; and the two points lying on the same side of X, and such that the azimuth of $S_{\circ}$ taken at $S_{\circ\circ}$ is acute, and the azimuth of $S_{\circ\circ}$ taken at $S_{\circ}$ is also acute but greater than the other, and approaching very nearly to 90° owing to the earth's small ellipticity.

24. From the triangles $S_{\prime\prime}PD_{\prime}$, $S_{\prime}PD_{\prime\prime}$, we have—

$$\sin L' = \frac{\sin z_{\prime\prime} \sin A_{\prime\prime}}{\sin \omega}$$
$$\sin L'' = \frac{\sin z_{\prime} \sin A_{\prime}}{\sin \omega} \tag{76}$$

$$\cos L' = \cos z_{\prime\prime} \cos l'' + \sin z_{\prime\prime} \sin l'' \cos A_{\prime\prime}$$
$$\cos L'' = \cos z_{\prime} \cos l' + \sin z_{\prime} \sin l' \cos A_{\prime} \tag{77}$$

$$\cot L' = \frac{\cot A_{\prime\prime} \sin \omega + \cos l'' \cos \omega}{\sin l''}$$
$$\cot L'' = \frac{\cot A_{\prime} \sin \omega + \cos l' \cos \omega}{\sin l'} \tag{78}$$

And since L′ and L″ are the circular measures of the angles between the lines $S_{\circ}Z_{\circ\circ}$, $S_{\circ\circ}Z_{\circ}$, and the polar axis, we have evidently—

$$\cot L' = e^2 \cdot \frac{R_{\prime\prime} \sin l_{\prime\prime}}{R_{\prime} \cos l_{\prime}} + (1 - e^2) \tan l_{\prime}$$
$$\cot L'' = e^2 \cdot \frac{R_{\prime} \sin l_{\prime}}{R_{\prime\prime} \cos l_{\prime\prime}} + (1 - e^2) \tan l_{\prime\prime} \tag{79}$$

25. By letting fall perpendiculars from $Z_{\circ\circ}$, $Z_{\circ}$, on the

normals $R_{,}$, $R_{,,}$, we easily find the following expressions for $\delta_{,}$ and $\delta_{,,}$—

$$\left.\begin{aligned} \tan\delta_{,} &= \frac{e^2 (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \cos l_{,}}{R_{,} - e^2 (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \sin l_{,}} \\ \tan\delta_{,,} &= \frac{e^2 (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \cos l_{,,}}{R_{,,} + e^2 (R_{,} \sin l_{,} - R_{,,} \sin l_{,,}) \sin l_{,,}} \end{aligned}\right\} \quad (80)$$

And from the plane triangles whose bases are $Z_{o}Z_{oo}$, and vertices S_{o}, S_{oo}, we have—

$$\left.\begin{aligned} \sin\delta_{,} &= \frac{e^2 (R_{,} \cos l' - R_{,,} \cos l'') \sin L'}{R_{,}} \\ \sin\delta_{,,} &= \frac{e^2 (R_{,} \cos l' - R_{,,} \cos l'') \sin L''}{R_{,,}} \end{aligned}\right\} \quad (81)$$

Again, from the triangles $S_{o}S_{oo}Z_{o}$, $S_{o}S_{oo}Z_{oo}$, it is evident that—

$$\frac{R_{,}}{Q_{,,}} = \frac{\cos(z_{,} - \alpha_{,})}{\cos \alpha_{,}}; \qquad \frac{R_{,,}}{Q_{,}} = \frac{\cos(z_{,,} - \alpha_{,,})}{\cos \alpha_{,,}}; \quad (81\text{A})$$

and, to 8 places of decimals in their logarithms, we have—

$$\frac{R_{,}}{Q_{,,}} = \frac{R_{,,}}{Q_{,}} = 1. \quad (81\text{B})$$

Hence, from the triangles $Z_{o}Z_{oo}S_{o}$, $Z_{o}Z_{oo}S_{oo}$, we have the relations—

$$\frac{\sin L'}{\sin l'} = \frac{R_{,}}{R_{,,}}; \qquad \frac{\sin L''}{\sin l''} = \frac{R_{,,}}{R_{,}}.$$

such that their logs. are the same to 7 places of decimals. And if in the first and second of (81) we substitute for $\frac{R_{,}}{R_{,,}}$, and $\frac{R_{,,}}{R_{,}}$ the above equivalents, we have with an accuracy to at least 7 places of decimals in their logs.—

$$\left.\begin{aligned} \sin\delta_{,} &= e^2 (\sin L' \cos l' - \cos l'' \sin l') \\ \sin\delta_{,,} &= e^2 (\cos l' \sin l'' - \sin L'' \cos l'') \end{aligned}\right\} \quad (82)$$

which we may write in the forms—

$$\sin\delta_{,} = e^2 \left\{ -\cos l'' \sin (L' - \delta_{,}) + \sin L' \cos (L' - \delta_{,}) \right\}$$

$$\sin\delta_{,,} = e^2 \left\{ \cos l' \sin (L'' + \delta_{,,}) - \sin L'' \cos (L'' + \delta_{,,}) \right\}$$

And if we expand these and regard $\cos\delta_{,} = \cos\delta_{,,} = 1$ (which we can do since $\delta_{,}$ or $\delta_{,,}$ is always less than 20″) we easily find—

$$\sin\delta_{,} = \frac{e^2 \cdot (\cos L' - \cos l'') \sin L'}{(1 - e^2) + e^2 (\cos L' - \cos l'') \cos L'}$$

$$\sin \delta_{\prime\prime} = \frac{e^2 \cdot (\cos l' - \cos L'') \sin L''}{(1 - e^2) - e^2 (\cos l' - \cos L'') \cos L''}$$

which we may write in the forms—

$$\sin \delta_{\prime} = \frac{2 \cdot e^2 \cdot \sin \frac{1}{2} (l'' + L') \sin \frac{1}{2} (l'' - L') \sin L'}{(1 - e^2) + 2 \cdot e^2 \cdot \sin \frac{1}{2} (l'' + L') \sin \frac{1}{2} (l'' - L') \cos L'} \tag{83}$$

$$\sin \delta_{\prime\prime} = \frac{2 \cdot e^2 \cdot \sin \frac{1}{2} (L'' + l') \sin \frac{1}{2} (L'' - l') \sin L''}{(1 - e^2) - 2 \cdot e^2 \cdot \sin \frac{1}{2} (L'' + l') \sin \frac{1}{2} (L'' - l') \cos L''}$$

(to be used when extreme accuracy is desired.)

Hence evidently (since $\delta_{\prime}$ or $\delta_{\prime\prime}$ is always less than 20 seconds) we have—

$$\sin \delta_{\prime} = 2 \left(\frac{e^2}{1 - e^2}\right) \sin L' \sin \tfrac{1}{2} (l'' + L') \sin \tfrac{1}{2} (l'' - L') \tag{84}$$

$$\sin \delta_{\prime\prime} = 2 \left(\frac{e^2}{1 - e^2}\right) \sin L'' \sin \tfrac{1}{2} (L'' + l') \sin \tfrac{1}{2} (L'' - l')$$

giving $\delta_{\prime}$ in excess, and $\delta_{\prime\prime}$ too small. However, in all ordinary cases, they give values of $\delta_{\prime}$, $\delta_{\prime\prime\prime}$, correct to $\frac{1}{1000}$ part of one second. And since—

$$\sin \tfrac{1}{2} (l'' + L') \sin \tfrac{1}{2} (l'' - L') = \sin (D_{\prime} - A_{\prime\prime}) \cdot \frac{\sin^2 \frac{1}{2} z_{\prime\prime}}{\sin \omega}$$

$$= \tfrac{1}{2} \cdot \sin (D_{\prime} - A_{\prime\prime}) \tan \tfrac{1}{2} z_{\prime\prime} \cdot \frac{\sin L'}{\sin A_{\prime\prime}}$$

$$\sin \tfrac{1}{2} (L'' + l') \sin \tfrac{1}{2} (L'' - l') = \sin (A_{\prime} - D_{\prime\prime}) \cdot \frac{\sin^2 \frac{1}{2} z_{\prime}}{\sin \omega}$$

$$= \tfrac{1}{2} \cdot \sin (A_{\prime} - D_{\prime\prime}) \tan \tfrac{1}{2} z_{\prime} \cdot \frac{\sin L''}{\sin A_{\prime}}$$

Therefore we have the equally approximate relations—

$$\sin \delta_{\prime} = 2 \left(\frac{e^2}{1 - e^2}\right) \sin L' \cdot \frac{\sin (D_{\prime} - A_{\prime\prime})}{\sin \omega} \cdot \sin^2 \tfrac{1}{2} z_{\prime\prime}$$

$$= \left(\frac{e^2}{1 - e^2}\right) \sin^2 L' \cdot \frac{\sin (D_{\prime} - A_{\prime\prime})}{\sin A_{\prime\prime}} \tan \tfrac{1}{2} z_{\prime\prime} \tag{85}$$

$$= 2 \left(\frac{e^2}{1 - e^2}\right) \sin l'' \cdot \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin D_{\prime} \sin \omega} \cdot \sin^2 \tfrac{1}{2} z_{\prime\prime}$$

$$= \left(\frac{e^2}{1 - e^2}\right) \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin^2 \omega} \cdot \sin^2 z_{\prime\prime} \cdot \tan \tfrac{1}{2} z_{\prime\prime}$$

$$= \left(\frac{e^2}{1 - e^2}\right) \sin^2 l'' \cdot \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin^2 D_{\prime}} \cdot \tan \tfrac{1}{2} z_{\prime\prime}$$

$$\sin \delta_{\prime\prime} = 2\left(\frac{e^2}{1-e^2}\right) \sin L'' \frac{\sin (A_{\prime} - D_{\prime\prime})}{\sin \omega} \cdot \sin^2 \tfrac{1}{2} z_{\prime}$$

$$= \left(\frac{e^2}{1-e^2}\right) \sin^2 L'' \cdot \frac{\sin (A_{\prime} - D_{\prime\prime})}{\sin A_{\prime}} \cdot \tan \tfrac{1}{2} z_{\prime}$$

$$= 2\left(\frac{e^2}{1-e^2}\right) \sin l' \cdot \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin D_{\prime\prime} \sin \omega} \cdot \sin^2 \tfrac{1}{2} z_{\prime} \qquad (86)$$

$$= \left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin^2 \omega} \sin^2 z_{\prime} \cdot \tan \tfrac{1}{2} z_{\prime}$$

$$= \left(\frac{e^2}{1-e^2}\right) \sin^2 l' \cdot \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin^2 D_{\prime\prime}} \tan \tfrac{1}{2} z_{\prime}$$

And since the arcs $z_{\prime}$, $z_{\prime\prime}$, do not exceed 1° in the usual cases of trigonometrical surveys, we have, with sufficient accuracy for some purposes—

$$\delta_{\prime} = \left(\frac{e^2}{1-e^2}\right) \cdot \sin L' \cdot \sin \tfrac{1}{2} (l'' + L') \cdot (l'' - L')$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \cdot \sin L' \cdot \frac{\sin (D_{\prime} - A_{\prime\prime})}{\sin \omega} \cdot z^2_{\prime\prime} \cdot \sin 1''$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \cdot \sin^2 L' \frac{\sin (D_{\prime} - A_{\prime\prime})}{\sin A_{\prime\prime}} \cdot z_{\prime\prime}$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin D_{\prime} \sin \omega} \cdot \sin l'' \cdot z^2_{\prime\prime} \cdot \sin 1'' \qquad (87)$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin^2 \omega} \cdot \sin^2 z_{\prime\prime} \cdot z_{\prime\prime}$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime\prime} \sin (D_{\prime} - A_{\prime\prime})}{\sin^2 D_{\prime}} \cdot \sin^2 l'' \cdot z_{\prime\prime}$$

$$\delta_{\prime\prime} = \left(\frac{e^2}{1-e^2}\right) \cdot \sin L'' \cdot \sin \tfrac{1}{2} (L'' + l') \cdot (L'' - l')$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \cdot \sin L'' \cdot \frac{\sin (A_{\prime} - D_{\prime\prime})}{\sin \omega} \cdot z^2_{\prime} \cdot \sin 1''$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \cdot \sin^2 L'' \cdot \frac{\sin (A_{\prime} - D_{\prime\prime})}{\sin A_{\prime}} \cdot z_{\prime}$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin D_{\prime\prime} \sin \omega} \cdot \sin l' \cdot z^2_{\prime} \cdot \sin 1'' \qquad (88)$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin^2 \omega} \cdot z^3_{\prime} \cdot \sin^2 1''$$

$$= \tfrac{1}{2}\left(\frac{e^2}{1-e^2}\right) \frac{\sin A_{\prime} \sin (A_{\prime} - D_{\prime\prime})}{\sin^2 D_{\prime\prime}} \cdot \sin^2 l' \cdot z_{\prime}$$

☞ Referring to the approximate relation—

$$\frac{\sin l'}{\sin L'} = \frac{\sin L''}{\sin l''}$$

made use of in arriving at the preceding values of $\delta_{,}$, $\delta_{,,}$, it may be proper to observe that we must not always use it as if it were rigorously true. If so used we should, as a consequence, have—

$$\frac{\sin A_{,}}{\sin D_{,}} = \frac{\sin D_{,,}}{\sin A_{,,}}$$

and therefore the first side of this equation always less than unity, which we know to be absurd. Hence we perceive that the adoption of the above approximate relation is equivalent to assuming that between the limits of the possible values of $A_{,}$ from the state in which $A_{,} = D_{,,}$ to that in which $A_{,} = V$, we have $\sin D_{,} = \sin A_{,}$, and $\sin A_{,,} = \sin D_{,,}$ so nearly true that their logarithms are the same to 7 places of decimals. However, we will now shew how those small angular differences can be computed.

26. It is evident that the amount by which the angle $A_{,,}$ exceeds $D_{,,}$ is truly expressed by the spherical excess of the small triangle $S_{,}S_{,,}D_{,,}$. It is also evident that the amount by which the angle $D_{,}$ exceeds $A_{,}$ is expressed by the spherical excess of the small triangle $S_{,}S_{,,}D_{,}$. Hence (see formula 4, page 158, Serrets', &c.)—

$$\begin{aligned}
\cot \tfrac{1}{2} A_{,,} &= \cot \tfrac{1}{2} D_{,,} \cdot \frac{\cos \frac{1}{2}(z_{,} + \delta_{,,})}{\cos \frac{1}{2}(z_{,} - \delta_{,,})} \\
\tan \tfrac{1}{2} A_{,,} &= \tan \tfrac{1}{2} D_{,,} \cdot \frac{\cos \frac{1}{2}(z_{,} - \delta_{,,})}{\cos \frac{1}{2}(z_{,} + \delta_{,,})} \\
\tan \tfrac{1}{2} A_{,} &= \tan \tfrac{1}{2} D_{,} \cdot \frac{\cos \frac{1}{2}(z_{,,} + \delta_{,})}{\cos \frac{1}{2}(z_{,,} - \delta_{,})} \\
\cot \tfrac{1}{2} A_{,} &= \cot \tfrac{1}{2} D_{,} \cdot \frac{\cos \frac{1}{2}(z_{,,} - \delta_{,})}{\cos \frac{1}{2}(z_{,,} + \delta_{,})}
\end{aligned} \qquad (89)$$

We have also (see formula 3, page 158, of Serrets' Trigonometry) rigorously—

$$\begin{aligned}
\tan \tfrac{1}{2}(A_{,,} - D_{,,}) &= \frac{\tan \frac{1}{2} z_{,} \tan \frac{1}{2} \delta_{,,} \sin D_{,,}}{1 - \tan \frac{1}{2} z_{,} \tan \frac{1}{2} \delta_{,,} \cos D_{,,}} \\
\tan \tfrac{1}{2}(D_{,} - A_{,}) &= \frac{\tan \frac{1}{2} z_{,,} \tan \frac{1}{2} \delta_{,} \sin D_{,}}{1 + \tan \frac{1}{2} z_{,,} \tan \frac{1}{2} \delta_{,} \cos D_{,}}
\end{aligned} \qquad (90)$$

And the angles $\frac{1}{2}(A_{,,} - D_{,,})$, $\frac{1}{2}(D_{,} - A_{,})$, being but fractions of a second; and the values of $\tan \frac{1}{2} z_{,} \cdot \tan \frac{1}{2} \delta_{,,}$

$\cos D_{,,}$ and $\tan \frac{1}{2} z_{,,} \cdot \tan \frac{1}{2} \delta_{,} \cdot \cos D_{,}$ being so extremely small, it is evident we can find the values of the angles $A_{,,}$ and $A_{,}$ to the $\frac{1}{10000}$ part of a second by means of the ameliorated formulæ—

$$\begin{aligned} \tan \tfrac{1}{2} (A_{,,} - D_{,,}) &= \sin D_{,,} \tan \tfrac{1}{2} z_{,} \cdot \tan \tfrac{1}{2} \delta_{,,} \\ \tan \tfrac{1}{2} (D_{,} - A_{,}) &= \sin D_{,} \tan \tfrac{1}{2} z_{,,} \cdot \tan \tfrac{1}{2} \delta_{,} \end{aligned} \quad (91)$$

We can also arrive at these in the following manner—

From formula (1), implicating spherical excess, on page 157 of Serrets' Trigonometry, we have—(since in actual practice of surveying the logs. of $\cos \frac{1}{2} \nu$, $\cos \frac{1}{2} z_{,}$, $\cos \frac{1}{2} z_{,,}$ are the same to 6 or 7 places of decimals)—

$$\begin{aligned} \sin \tfrac{1}{2} (A_{,,} - D_{,,}) &= \sin D_{,,} \cdot \tan \tfrac{1}{2} z_{,} \cdot \sin \tfrac{1}{2} \delta_{,,} \\ \sin \tfrac{1}{2} (D_{,} - A_{,}) &= \sin D_{,} \cdot \tan \tfrac{1}{2} z_{,,} \cdot \sin \tfrac{1}{2} \delta_{,} \end{aligned} \quad (92)$$

∴ also

$$\begin{aligned} A_{,,} - D_{,,} &= \sin D_{,,} \tan \tfrac{1}{2} z_{,} \cdot \delta_{,,} \\ D_{,} - A_{,} &= \sin D_{,} \tan \tfrac{1}{2} z_{,,} \cdot \delta_{,} \end{aligned} \quad (93)$$

or,

$$\begin{aligned} A_{,,} - D_{,,} &= \tfrac{1}{2} \cdot z_{,} \cdot \delta_{,,} \cdot \sin 1'' \cdot \sin D_{,,} \\ D_{,} - A_{,} &= \tfrac{1}{2} \cdot z_{,,} \cdot \delta_{,} \cdot \sin 1'' \cdot \sin D_{,} \end{aligned}$$

And from these and formulæ (87) and (88), we easily find—

$$\begin{aligned} A_{,,} - D_{,,} &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin l' \cdot \sin L'' \sin (A_{,} - D_{,,}) \cdot z_{,}^2 \times \sin 1'' \\ &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin^2 l' \cdot \frac{\sin A_{,} \sin (A_{,} - D_{,,})}{\sin D_{,,}} \cdot z_{,}^2 \times \sin 1'' \\ &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin l' \cdot \frac{\sin A_{,} \sin (A_{,} - D_{,,})}{\sin \omega} \cdot z_{,}^3 \times \sin^2 1'' \end{aligned}$$

$$\begin{aligned} D_{,} - A_{,} &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin l'' \cdot \sin L' \cdot \sin (D_{,} - A_{,,}) \cdot z_{,,}^2 \times \sin 1'' \\ &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin^2 l'' \cdot \frac{\sin A_{,,} \sin (D_{,} - A_{,,})}{\sin D_{,}} \cdot z_{,,}^2 \times \sin 1'' \\ &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \sin l'' \cdot \frac{\sin A_{,,} \sin (D_{,} - A_{,,})}{\sin \omega} \cdot z_{,,}^3 \times \sin^2 1'' \end{aligned}$$

☞ In the "Account of the Principal Triangulation of Great Britain and Ireland" (see pages 248, 249, formulæ 32 and 36), the following erroneous expressions are given—

$$\begin{aligned} D_{,,} - A_{,,} &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \cos^2 l_{,} \sin 2A_{,} \cdot z^2_{,} \times \sin 1'' \\ D_{,} - A_{,} &= \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \cos^2 l_{,,} \sin 2A_{,,} \cdot z^2_{,,} \times \sin 1'' \end{aligned} \quad (96)$$

with respect to which we may observe—

1°. From them we should infer that $D_{,,} - A_{,,}$ and $D_{,} - A_{,}$ have finite values when the latitudes of the stations are equal; but we know, in any such case, that the angles $D_{\prime\prime}$, $A_{\prime\prime}$, $D_{\prime}$, $A_{\prime}$ are equal.

2°. From the first of the equations we should infer that $A_{\prime\prime}$ is less than $D_{\prime\prime}$ when $A_{\prime}$ is acute; but we know that $A_{\prime\prime}$ must be always greater than $D_{\prime\prime}$, when $l_{\prime}$ is greater than $l_{\prime\prime}$, or when $A_{\prime}$ is greater than $A_{\prime\prime}$.

3°. In the example 1 worked out in this paper, we have, by using correct formulæ—

$$A_{\prime\prime} - D_{\prime\prime} = 0'' \cdot 1334; \quad D_{\prime} - A_{\prime} = 0'' \cdot 1334.$$

But if we were to use the above erroneous formulæ, we would find the values—

$$A_{\prime\prime} - D_{\prime\prime} = 0'' \cdot 1315; \quad D_{\prime} - A_{\prime} = 0'' \cdot 1352.$$

☞ On page 676 the formula 96 is misprinted: $\frac{1}{\sin 1''}$ being there used instead of $\sin 1''$.

27. From (46) and (47) it is easy to deduce the following expression—

$$\sin \tfrac{1}{2} \nu = \frac{\sqrt{\cos \tfrac{1}{2} (A_{\prime} + A_{\prime\prime} + x) \cos \tfrac{1}{2} (A_{\prime} + A_{\prime\prime} - x)}}{\cos \tfrac{1}{2} (A_{\prime} + A_{\prime\prime})}$$

in which the angle x is found from—

$$\sin \tfrac{1}{2} x = \sin \tfrac{1}{2} (l_{\prime} + l_{\prime\prime}) \cdot \sin \tfrac{1}{2} \omega.$$

28. The perpendicular from Z_{oo} to the line $S_{oo}Z_o$ is equal $Z_oZ_{oo} \cdot \sin L''$; and ∴ it is evident that the perpendicular from Z_{oo} on the normal-chordal plane $S_oS_{oo}Z_o$ is equal to $Z_oZ_{oo} \cdot \sin L'' \cdot \sin D_{\prime\prime}$. But the perpendicular from Z_{oo} on the chord S_oSS_{oo} is evidently equal to $R_{\prime\prime} \cdot \cos \alpha_{\prime\prime}$. Hence, obviously—

$$\sin \triangle = \frac{Z_oZ_{oo} \cdot \sin L'' \cdot \sin D_{\prime\prime}}{R_{\prime\prime} \cdot \cos \alpha_{\prime\prime}}$$

But,

$$Z_oZ_{oo} = e^2 (R_{\prime} \sin l_{\prime} - R_{\prime\prime} \sin l_{\prime\prime}); \quad \sin L'' \sin D_{\prime\prime} = \cos l_{\prime} \sin A_{\prime};$$

and

$$\cos \alpha_{\prime\prime} = \frac{R_{\prime} \cos l_{\prime} \sin \omega}{k \cdot \sin A_{\prime\prime}}$$

Hence we have—

$$\sin \triangle = e^2 \cdot k \cdot \frac{\sin A_{\prime} \sin A_{\prime\prime}}{\sin \omega} \cdot \left(\frac{\sin l_{\prime}}{R_{\prime\prime}} - \frac{\sin l_{\prime\prime}}{R_{\prime}}\right) \qquad (98)$$

$$\sin \triangle = k \cdot \frac{R^2_{\prime} - R^2_{\prime\prime}}{R_{\prime} \cdot R_{\prime\prime}} \cdot \frac{\sin A_{\prime} \sin A_{\prime\prime}}{\sin \omega} \cdot (R_{\prime} \sin l' + R_{\prime\prime} \sin l_{\prime\prime})^{-1}$$

$$\sin \triangle = \frac{(R^2_{\prime} - R^2_{\prime\prime})^{\frac{1}{2}}}{R_{\prime} \sin l_{\prime} + R_{\prime\prime} \sin l_{\prime\prime}} \cdot (\cos^2 l_{\prime\prime} \sin^2 A_{\prime\prime} - \cos^2 l_{\prime} \sin^2 A_{\prime})$$

These expressions are rigorously true, and can be used in other investigations.

We have also from the triangles $IS_{,}D_{,,}$, $IS_{,,}D_{,,,}$—

$$\sin \triangle = \frac{\sin \delta_{,} \cdot \sin D_{,}}{\cos \frac{1}{2} \Sigma} = \frac{\sin \delta_{,,} \sin D_{,,}}{\cos \frac{1}{2} \Sigma} \qquad (101)$$

☞ In the "Account of the Principal Triangulation of Great Britain and Ireland," the following expressions are given—

$$\triangle = e^2 \cdot \sin 2 A_{,} \cdot \cos^2 (l_{,} + l_{,,}) \cdot \tfrac{1}{2} \Sigma$$
$$\triangle = e^2 \cdot \sin 2 A_{,,} \cdot \cos^2 (l_{,} + l_{,,}) \cdot \tfrac{1}{2} \Sigma \qquad (102)$$

That this formula is erroneous is easily seen: for independent of the oversight committed in assuming that $\sin 2 A_{,}$ is equal to $\sin 2 A_{,,}$, we know that any expression representing $\triangle$ must vanish when the latitudes $l_{,}$, $l_{,,}$ are equal; and this is not the case with formulæ (102).

29. When the stations S_{o}, S_{oo}, are mutually visible (not more than 100 miles apart), it is evident that if from the middle point of the arc ν we conceive perpendicular arcs drawn to the circles $S_{,}D_{,,}$, $S_{,,}D_{,}$, they will form two right angled spherical triangles (having vertices at $S_{,}$ and $S_{,,}$), which may be considered equals in all respects. It is evident that two of the sides of either of these triangles are equals to $\frac{1}{2} \nu$ and $\frac{1}{2} \Sigma$, and that the third side of either may be regarded as equal to $\frac{1}{2} \triangle$.

From this relation connecting the angle between the normals, the angle between the normal-chordal planes, and the circular measure of the geodesic arc between the stations, we have—

$$\cos \tfrac{1}{2} \nu = \cos \tfrac{1}{2} \triangle \cdot \cos \tfrac{1}{2} \Sigma \qquad (103)$$

$$\sin \tfrac{1}{2} \triangle = \sin \tfrac{1}{2} \nu \cdot \sin \Omega \qquad (104)$$

$$\tan \tfrac{1}{2} \triangle = \sin \tfrac{1}{2} \Sigma \cdot \tan \Omega \qquad (105)$$

$$\tan \tfrac{1}{2} \Sigma = \tan \tfrac{1}{2} \nu \cdot \cos \Omega \qquad (106)$$

simple relations which will be found very useful in practical work of trigonometrical surveys.

30. The following expressions for the cosines, sines, and tangents of the angles of depression of the chord are rigorous with respect to any two stations on the earth's spheroidal surface; and the easy methods by which they have been deduced (from what has been already done) are omitted, as they can present no difficulty to the reader.

$$\cos a_{,} = \frac{R_{,,} \cos l_{,,} \sin \omega}{k \cdot \sin A_{,}}$$

$$\cos a_{,,} = \frac{R_{,} \cos l_{,} \sin \omega}{k \cdot \sin A_{,,}} \qquad (107)$$

$$\sin a_{,} = \frac{R_{,} \cos l_{,} - R_{,,} \cos l_{,,} (\tan l_{,} \cot A_{,} \sin \omega + \cos \omega)}{k \cdot \cos l_{,}}$$

$$\sin a_{,,} = \frac{R_{,,} \cos l_{,,} - R_{,} \cos l_{,} (\tan_{,,} \cot A_{,,} \sin \omega + \cos \omega)}{k \cdot \cos l_{,,}} \qquad (108)$$

$$\sin a_{,} = \frac{R_{,}R_{,,} (\cos l_{,} \cos l_{,,} \cos \omega + (1-e^2) \sin l_{,} \sin l_{,,}) - a^2}{k \cdot R_{,}}$$

$$\sin a_{,,} = \frac{R_{,}R_{,,} (\cos l_{,} \cos l_{,,} \cos \omega + (1-e^2) \sin l_{,} \sin l_{,,}) - a^2}{k \cdot R_{,,}} \qquad (109)$$

$$\tan a_{,} = \frac{R_{,} \sin A_{,}}{R_{,,} \cdot \cos l_{,,} \sin \omega} - \frac{\cot \omega \sin A_{,} + \sin l_{,,} \cos A_{,}}{\cos l_{,}}$$

$$\tan a_{,,} = \frac{R_{,,} \sin A_{,,}}{R_{,} \cos l_{,} \sin \omega} - \frac{\cot \omega \sin A_{,,} + \sin l_{,,} \cos A_{,,}}{\cos l_{,,}} \qquad (110)$$

$$\tan a_{,} = \frac{\cos l_{,}}{\sin A_{,,}} \cdot \frac{R_{,,} \sin A_{,,} \cos A_{,} + R_{,} \cos A_{,,} \sin A_{,}}{R_{,,} \sin l_{,,} + R_{,} \sin l_{,}}$$

$$\tan a_{,,} = \frac{\cos l_{,,}}{\sin A_{,}} \cdot \frac{R_{,,} \sin A_{,,} \cos A_{,} + R_{,} \cos A_{,,} \sin A_{,}}{R_{,,} \sin l_{,,} + R_{,} \sin l_{,}} \qquad (111)$$

31. By equating the values of $\sin a_{,}$ given in (108), (109), we have an equation from which we can at once express $\cot A_{,}$ in terms of the two latitudes and the difference of longitude ω. And equating the values of $\sin a_{,,}$ given in (108), (109), we can express $\cot A_{,,}$ in terms of the two latitudes and difference of longitude. However, we can find other expressions for the cotangents of the azimuths, thus—

From the spherical triangles $S_{,}PD_{,,}$, $S_{,,}PD_{,}$, we have

$$\cot A_{,} = \frac{\cot L'' \cos l_{,} - \sin l_{,} \cos \omega}{\sin \omega}$$

$$\cot A_{,,} = \frac{\cot L' \cos l_{,,} - \sin l_{,,} \cos \omega}{\sin \omega}$$

And if in these we substitute the values of $\cot L''$, $\cot L'$, given in (79), we have—

$$\cot A_{,} = \frac{\frac{R_{,}}{R_{,,}} \cdot e^2 \sin l_{,} \cos l_{,} + (1-e^2) \sin l_{,,} \cos l_{,} - \sin l_{,} \cos l_{,,} \cos \omega}{\cos l_{,,} \sin \omega}$$

$$\cot A_{,,} = \frac{\frac{R_{,,}}{R_{,}} \cdot e^2 \sin l_{,,} \cos l_{,,} + (1-e^2) \sin l_{,} \cos l_{,,} - \sin l_{,,} \cos l_{,} \cos \omega}{\cos l_{,} \sin \omega} \qquad (112)$$

These have been arrived at by other means in the "Account of the Principal Triangulation of Great Britain and Ireland." Moreover, from the spherical triangle $S_{,}PS_{,,}$ we have—

$$\cot A_{\circ} = \frac{\sin l_{,,} \cos l_{,} - \sin l_{,} \cos l_{,,} \cos \omega}{\cos l_{,,} \sin \omega}$$

$$\cot A_{\circ\circ} = \frac{\sin l_{,} \cos l_{,,} - \sin l_{,,} \cos l_{,} \cos \omega}{\cos l_{,} \sin \omega}$$

$$\therefore \cot A_{,} - \cot A_{\circ} = \left(\frac{R_{,}}{R_{,,}} \sin l_{,} - \sin l_{,,}\right) \cdot \frac{e^2 \cdot \cos l_{,}}{\cos l_{,,} \sin \omega} \quad (113)$$

$$\cot A_{,,} - \cot A_{\circ\circ} = \left(\frac{R_{,,}}{R_{,}} \sin l_{,,} - \sin l_{,}\right) \cdot \frac{e^2 \cdot \cos l_{,,}}{\cos l_{,} \sin \omega}$$

These also are given in the "Account of the Principal Triangulation of Great Britain and Ireland" (see page 231 of that work).

32. From (60) it is evident that for any pair of mutually visible stations, we have—

$$k = \frac{R_{,} \cos l_{,} \sin \omega}{\sin A_{,,} \cos \frac{1}{2} \Sigma}$$

$$k = \frac{R_{,,} \cos l_{,,} \sin \omega}{\sin A_{,} \cos \frac{1}{2} \Sigma} \quad (114)$$

$$k = \frac{R_{,}R_{,,}}{(R^2_{,} - R^2_{,,})^{\frac{1}{2}}} \cdot \frac{\sin \omega}{\sin A_{,} \sin A_{,,}} \cdot (\cos^2 l_{,,} \sin^2 A_{,,} - \cos^2 l_{,} \sin^2 A_{,})$$

the last of which is rigorously accurate for any two stations on the earth's spheroidal surface, and a direct expression in terms of the two latitudes and difference of longitude; but it assumes the form $\frac{0}{0}$ when the latitudes $l_{,}$, $l_{,,}$ are equal.

33. From $\frac{\sin^2 \alpha_{,}}{\sin^2 \alpha_{,,}} = \frac{R^2_{,,}}{R^2_{,}} = \frac{1 - e^2 \sin^2 l_{,}}{1 - e^2 \sin^2 l_{,,}}$, we have the rigorous formulæ—

$$e^2 = \frac{\sin^2 \alpha_{,,} - \sin^2 \alpha_{,}}{\sin^2 l_{,} \sin^2 \alpha_{,,} - \sin^2 l_{,,} \sin^2 \alpha_{,}} \quad (115)$$

$$\frac{b^2}{a^2} = \frac{\cos^2 l_{,} \sin^2 \alpha_{,,} - \cos^2 l_{,,} \sin^2 \alpha_{,}}{\sin^2 l_{,} \sin^2 \alpha_{,,} - \sin^2 l_{,,} \sin^2 \alpha_{,}} \quad (116)$$

applying to any two stations whatever on the earth's spheroidal surface.

From (53) we have—

$$e^2 = \frac{\sin^2 A_{,,} \sec^2 l_{,} - \sin^2 A_{,} \sec^2 l_{,,}}{\sin^2 A_{,,} \tan^2 l_{,} - \sin^2 A_{,} \tan^2 l_{,,}} \quad (117)$$

$$\frac{b^2}{a^2} = \frac{\sin^2 A_{,} - \sin^2 A_{,,}}{\sin^2 A_{,,} \tan^2 l_{,} - \sin^2 A_{,} \tan^2 l_{,,}} \quad (118)$$

(Holding true to at least 8 places of decimals in their logarithms.)

The expressions for e^2 and $\frac{b^2}{a^2}$ in 115, 116, 117, 118, assume the form $\frac{0}{0}$ when the latitudes of the stations are equal. If the latitudes and mutual azimuths of *numerous pairs of suitable stations* be carefully found from actual observation with good instruments, &c., it is obvious that 117 and 118 will enable us to find the most probably correct or suitable value for the earth's eccentricity in the locality of the survey. And the great importance of having such a value of e will be obvious from the examples worked out in the sequel.

We can easily find other expressions for e^2 from 78 and 79, by substituting in (79) the values of $\frac{R_{\prime\prime}}{R_{\prime}}$ and $\frac{R_{\prime}}{R_{\prime\prime}}$ given in 51.

34. It may be seen, from a glance at the figure, that when the two stations have not the same latitude, a difference in the heights of the stations (with respect to the earth's spheroidal surface) will introduce errors into the observed values of the azimuths $A_{\prime}$, $A_{\prime\prime}$ and other azimuthal readings.

1°. It is evident that according as the station S_{00} is higher or lower than the station S_0 by the length h, so will the observed azimuth $A_{\prime}$ be too great or too small by an angle μ which the length expressed by $h \times \sin \triangle$ subtends at the distance s. And according as the station S_0 is higher or lower than the station S_{00} by the length h, so will the observed azimuth $A_{\prime\prime}$ be too small or too great by an angle μ which the length expressed by $h \times \sin \triangle$ subtends at the distance s.

2°. It is ∴ obvious that when the station S_0 is higher than the station S_{00} then will the azimuths $A_{\prime}$ and $A_{\prime\prime}$, as found by direct observation, be too small; and when the station S_{00} is higher than the station S_0 then will the azimuths $A_{\prime}$ and $A_{\prime\prime}$, as found by direct observation, be too large.

☞ To find the error of correction μ, we have—

$$\mu = \frac{h}{s} \cdot \triangle$$

Now, in an example given in the sequel, we have $s = 513{,}906$ feet, and $\triangle = 10''{\cdot}85$. And according as we suppose the station S_0 to be higher or lower than the station S_{00} by the length $h = 10{,}000$ feet, so will each of the azimuths $A_{\prime}$, $A_{\prime\prime}$, be too small or too great by

$$\mu = 0'' \cdot 211$$

35. We will now consider how the magnitude of the angle $\triangle$ varies when the stations S_o, S_{oo}, are supposed to be situated on two fixed parallels of latitude, and at such distances asunder as may or can occur in trigonometrical surveying.

From equation 100 we at once perceive that when the latitudes $l_{,}$ $l_{,,}$ are constants, the angle $\triangle$ between the normal-chordal planes increases or decreases according as

$\cos^2 l_{,,} \sin^2 A_{,,} - \cos^2 l_{,} \sin^2 A_{,}$ increases or decreases.

Or, if in this we substitute for $\sin^2 A_{,}$ its equivalent as given by equation 50, then we know that $\triangle$ increases or decreases according as the expression

$\sin^2 A_{,,} \left(R^2_{,} \cos^2 l_{,,} - R^2_{,,} \cos^2 l_{,} \cdot \frac{\cos^2 a_{,,}}{\cos^2 a_{,}} \right)$ increases or decreases.

Now $A_{,,}$ being the necessarily acute and lesser azimuth, we know that $\sin^2 A_{,,}$ increases as the azimuth $A_{,,}$ increases. And, since $\frac{\sin a_{,,}}{\sin a_{,}} = \frac{R_{,}}{R_{,,}}$ is constant, and that $a_{,,}$ and $a_{,}$ increase or decrease according as the difference of longitude ω increases or decreases, it is evident that $\frac{1 - \sin^2 a_{,,}}{1 - \sin^2 a_{,}}$ or $\frac{\cos^2 a_{,,}}{\cos^2 a_{,}}$ *decreases* according as the difference of longitude *increases*; and $\therefore$ that $\triangle$ increases as ω and $A_{,,}$ increase up to that point at which the trace of the normal-chordal plane containing $R_{,,}$ touches the parallel of latitude on which S_o is situated.

36. Other new and useful formulæ can be easily derived from the figure. For instance, from the spherical triangles $S_{,}PI$, $S_{,,}PI$,

$$\begin{aligned} \cos\theta &= \sin a_{,} \sin l_{,} - \cos a_{,} \cos l_{,} \cos A_{,} \\ \cos\theta &= -\sin a_{,,} \sin l_{,,} + \cos a_{,,} \cos l_{,,} \cos A_{,,} \end{aligned} \qquad (119)$$

$$\therefore \sin a_{,} \sin l_{,} + \sin a_{,,} \sin l_{,,} = \cos a_{,} \cos l_{,} \cos A_{,} + \cos a_{,,} \cos l_{,,} \cos A_{,,} \qquad (120)$$

and hence with close approximation to absolute accuracy, we have

$$\tan a_{,} \sin l_{,} + \tan a_{,,} \sin l_{,,} = \cos l_{,} \cos A_{,} + \cos l_{,,} \cos A_{,,}$$

but

$$\frac{\tan a_{,}}{\tan a_{,,}} = \frac{\cos l_{,} \sin A_{,}}{\cos l_{,,} \sin A_{,,}}$$

And from these we easily find

$$\begin{aligned} \tan a_{,} &= \frac{\cos l_{,} \cos A_{,} + \cos l_{,,} \cos A_{,,}}{\cos l_{,} \sin l_{,} \sin A_{,} + \cos l_{,,} \sin l_{,,} \sin A_{,,}} \cdot \cos l_{,} \sin A_{,} \\ \tan a_{,,} &= \frac{\cos l_{,} \cos A_{,} + \cos l_{,,} \cos A_{,,}}{\cos l_{,} \sin l_{,} \sin A_{,} + \cos l_{,,} \sin l_{,,} \sin A_{,,}} \cdot \cos l_{,,} \sin A_{,,} \end{aligned} \qquad (121)$$

and $\therefore$

$$\tan \tfrac{1}{2} \Sigma = \frac{\cos l_{,} \cos A_{,} + \cos l_{,,} \cos A_{,,}}{\sin 2 l_{,} \sin A_{,} + \sin 2 l_{,,} \sin A_{,,}} \cdot 2 \sqrt{\cos l_{,} \cos l_{,,} \sin A_{,} \sin A_{,,}}$$

The expressions given for the tangents of the angles of depression of the geodesic chord in (110) and (111) implicate the assumed eccentricity of the earth, while the expressions (121) depend entirely on the observed latitudes and azimuths. If applied to the example 1 problem 1 given in the sequel (which may be regarded as an extreme case in trigonometrical surveying) it will be found that the resulting values of $\alpha_{,}$ and $\alpha_{,,}$ can be accurately determined to $\frac{1}{1000}$ part of one second,—their logs. holding true to 8 places of decimals.

By substituting in (111) the values $\frac{R_{,,}}{R_{,}}$ and $\frac{R_{,}}{R_{,,}}$ as given in (51), we easily rearrive at formulæ (121); and by like substitutions in (110), we easily find the following values for the tangents of the angles of depression of the chord — true to at least 8 places of decimals in their logs —

$$\tan \alpha_{,} = \frac{\sin A_{,,}}{\cos l_{,} \sin \omega} - \frac{\sin A_{,} \cot \omega + \cos A_{,} \sin l_{,}}{\cos l_{,}}$$

$$= \frac{\sin A_{,,}}{\cos l_{,} \sin \omega} - \cot z_{,}$$

$$\tan \alpha_{,,} = \frac{\sin A_{,}}{\cos l_{,,} \sin \omega} - \frac{\sin A_{,,} \cot \omega + \cos A_{,,} \sin l_{,,}}{\cos l_{,,}}$$

$$= \frac{\sin A_{,}}{\cos l_{,,} \sin \omega} - \cot z_{,,}$$

And when $\alpha_{,,}$ and $\alpha_{,}$ are found, we have $\Sigma = \alpha_{,,} + \alpha_{,}$.

However, there are other methods of finding approximate values of Σ, in terms of the latitudes, azimuths, and length of arc between the stations, &c.; but I defer their consideration for a future paper.

37. With respect to the figure it may be observed that if $F_{,}$ and $F_{,,}$ be the points in which the chordal plane $NS_{o}S_{oo}$ cuts the arcs $PS_{,}$, $PS_{,,}$, it is evident that the arc $PF_{,}$ is divided harmonically in $S_{,}$, $D_{,}$, and that the arc $PF_{,,}$ is divided harmonically in $D_{,,}$, $S_{,,}$. For the anharmonic ratio of the points $PF_{,}S_{,}D_{,}$ is the same as that of the pencil of straight lines $S_{o} \cdot (PF_{,}S_{,}D_{,})$, and $\therefore$ the same as that of the four points ∞, N, Z_{o}, Z_{oo}, in which ∞ represents the point at infinity in which the line $S_{o}P$ cuts the line $CZ_{o}Z_{oo}$, &c. Hence the spherical pencil $I \cdot (PF_{,}S_{,}D_{,})$ is harmonic.

Again, since $S_0F_{\prime}$, $S_0F_{\prime\prime}$, S_0O, are parallels to NS_0, NS_{00}, NM, it follows that the arc $F_{\prime}F_{\prime\prime}$ is bisected in O; and therefore (as arc IO is a quadrant) the arc IO is cut harmonically in $F_{\prime}$, $F_{\prime\prime}$; and the spherical pencil P · $(IOF_{\prime}F_{\prime\prime})$ is harmonic.

NOTATION.

When any number n of stations are to be simultaneously considered.

Let 1, 2, 3, ... , n, indicate stations on the earth's surface.

„ $l_1, l_2, l_3,$ . . . . . , l_n, indicate the latitudes at these stations.

„ $R_1, R_2, R_3,$ . . . . . , R_n, „ the normals terminating in polar axis.

„ $\omega_{12}, \omega_{23}, \omega_{34},$ . . . . . . „ the differences of longitude between the pairs of stations 1, 2; 2, 3; 3, 4; . . . .

Put A_{12}, A_{21}, for the azimuths of the stations 2, 1, as if observed from 1 and 2.

„ A_{23}, A_{32}, for the azimuths of the stations 3, 2, as if observed from 2 and 3.

„ ... ...

„ ... ...

„ α_{12}, α_{21}, for the angles of depression of the chord 1, 2, at the stations 1 and 2.

„ α_{23}, α_{32}, for the angles of depression of the chord 2, 3, at the stations 2 and 3.

„ ... ...

„ ... ...

„ k_{12}, k_{23}, k_{34}, for the chords 1, 2; 2, 3; 3, 4; of the spheroidal triangle 1, 2, 3.

„ Σ_{12}, Σ_{13}, Σ_{23}, for the spherical measures $\alpha_{12} + \alpha_{21}$; $\alpha_{13} + \alpha_{31}$; $\alpha_{23} + \alpha_{32}$; of the sides of the spheroidal triangle 1, 2, 3.

„ s_{12}, s_{13}, s_{23}, for the lengths of the sides 1, 2; 1, 3; 2, 3; of the spheroidal triangle 1, 2, 3.

1. For any n stations 1, 2, 3, ... $n - 1$, n, on the earth's spheroidal surface, we have the rigorously accurate equations

$$\frac{R_2}{R_{\prime}} = \frac{\sin \alpha_{12}}{\sin \alpha_{21}};\ \frac{R_3}{R_2} = \frac{\sin \alpha_{23}}{\sin \alpha_{32}};\ \cdots\cdots\cdots\ \frac{R_n}{R_{n-1}} = \frac{\sin \alpha_{n-1,n}}{\sin \alpha_{n,n-1}}$$

and ∴

$$\frac{R_n}{R_{\prime}} = \frac{\sin \alpha_{12} \cdot \sin \alpha_{23} \cdots\cdots\cdots \sin \alpha_{n-1,n}}{\sin \alpha_{21} \cdot \sin \alpha_{32} \cdots\cdots\cdots \sin \alpha_{n,n-1}} \qquad (123)$$

And putting M to represent the reciprocal of the dexter of this equation, we easily find—

$$\sin^2 l_n = \frac{1}{e^2} - \left(\frac{1}{e^2} - \sin^2 l_{,}\right) \cdot M^2 \qquad (124)$$

an equation expressing the latitude of the n^{th} station in terms of the latitude of the 1st station and the sines of the angles of depression of the $n - 1$ chords joining the consecutive stations.

2. We have also the rigorously accurate relations

$$\frac{R_2 \cos l_2}{R_{,} \cos l_{,}} = \frac{\sin A_{12} \cos \alpha_{12}}{\sin A_{21} \cos \alpha_{21}}; \quad \frac{R_3 \cos l_3}{R_2 \cos l_2} = \frac{\sin A_{23} \cos \alpha_{23}}{\sin A_{32} \cos \alpha_{32}};$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots; \quad \ldots\ldots = \ldots\ldots\ldots\ldots;$$

and ∴

$$\frac{R_n \cos l_n}{R_{,} \cos l_{,}} = \frac{\sin A_{12} \sin A_{23} \ldots\ldots}{\sin A_{21} \sin A_{32} \ldots\ldots} \cdot \frac{\cos \alpha_{12} \cos \alpha_{23} \ldots\ldots}{\cos \alpha_{21} \cos \alpha_{32} \ldots\ldots} \qquad (125)$$

$$= \frac{\sqrt{(1 - e^2) \tan^2 l_{,} + 1}}{\sqrt{(1 - e^2) \tan^2 l_n + 1}}$$

and from this we easily find—

$$\tan^2 l_n = \left(\tan^2 l_{,} + \frac{1}{1 - e^2}\right) \cdot \left(\frac{\sin A_{21} \cdot \sin A_{32} \ldots\ldots}{\sin A_{12} \cdot \sin A_{23} \ldots\ldots}\right)^2 \cdot \left(\frac{\cos \alpha_{21} \cdot \cos \alpha_{32} \ldots\ldots}{\cos \alpha_{12} \cdot \cos \alpha_{23} \ldots\ldots}\right)^2 - \frac{1}{1 - e^2}$$

an equation expressing the latitude of the n^{th} station in terms of the latitude of the 1^{st} station, the azimuths, and the angles of depression of the chords connecting the stations.

3. And from (123) and (125) we have—

$$\frac{\cos l_n}{\cos l_{,}} = \frac{\sin A_{12} \cdot \sin A_{23} \ldots\ldots}{\sin A_{21} \cdot \sin A_{32} \ldots\ldots} \cdot \frac{\tan \alpha_{21} \cdot \tan \alpha_{32} \ldots\ldots}{\tan \alpha_{12} \cdot \tan \alpha_{23} \ldots\ldots} \qquad (127)$$

4. Let 1, 2, 3, ……….. $n - 1$, n, be any *odd* number of stations on the earth's spheroidal surface, such that none of the chords (12), (23), ……… $(n - 1, n)$, exceeds 100 miles in length. Then, from formula 49, it is evident we have the relations—

$$\frac{\tan (45^\circ - \frac{1}{2} l_{,})}{\tan (45^\circ - \frac{1}{2} l_3)} = \frac{\cos \frac{1}{2} (A_{12} + A_{21} + \omega_{12})}{\cos \frac{1}{2} (A_{12} + A_{21} - \omega_{12})} \div \frac{\cos \frac{1}{2} (A_{23} + A_{32} + \omega_{23})}{\cos \frac{1}{2} (A_{23} + A_{32} - \omega_{23})}$$

$$\frac{\tan\left(45^\circ - \frac{1}{2}\,l_3\right)}{\tan\left(45^\circ - \frac{1}{2}\,l_5\right)} = \frac{\cos\frac{1}{2}\left(A_{34} + A_{43} + \omega_{34}\right)}{\cos\frac{1}{2}\left(A_{34} + A_{43} - \omega_{34}\right)} \div \frac{\cos\frac{1}{2}\left(A_{45} + A_{54} + \omega_{45}\right)}{\cos\frac{1}{2}\left(A_{45} + A_{54} - \omega_{45}\right)} \quad (128)$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots \div \ldots\ldots\ldots\ldots$$

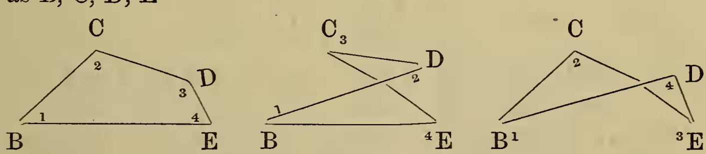
$$\frac{\tan\left(45^\circ - \frac{1}{2}\,l_{n-2}\right)}{\tan\left(45^\circ - \frac{1}{2}\,l_n\right)} = \frac{\cos\frac{1}{2}\left(\,.\;.\;.\;.\;.\,\right)}{\cos\frac{1}{2}\left(\,.\;.\;.\;.\;.\,\right)} \div \frac{\cos\frac{1}{2}\left(\,.\;.\;.\;.\;.\,\right)}{\cos\frac{1}{2}\left(\,.\;.\;.\;.\;.\,\right)}$$

And therefore we have—

$$\frac{\tan\left(45^\circ - \frac{1}{2}\,l_1\right)}{\tan\left(45^\circ - \frac{1}{2}\,l_n\right)} = \text{the product of the dexters of these } \frac{n-1}{2} \text{ equations,}$$

an equation from which we can at once express the latitude of the n^{th} station in terms of the latitude of the 1^{st} station and the azimuths and differences of longitudes.

Should the n^{th} station be coincident with the 1^{st} station, we must have the dexter of (129) equal to unity. This fact will be found to be of importance in case any *even* number of stations form the vertices of a closed geodesic polygon. For instance, if there be *four* mutually visible stations such as B, C, D, E—



then numbering the stations in the orders indicated in the above diagrams, we have—

$$\frac{\cos\frac{1}{2}\left(A_{12}+A_{21}+\omega_{12}\right)}{\cos\frac{1}{2}\left(A_{12}+A_{21}-\omega_{12}\right)} \cdot \frac{\cos\frac{1}{2}\left(A_{34}+A_{43}+\omega_{34}\right)}{\cos\frac{1}{2}\left(A_{34}+A_{43}-\omega_{34}\right)}$$

$$= \frac{\cos\frac{1}{2}\left(A_{23}+A_{32}+\omega_{23}\right)}{\cos\frac{1}{2}\left(A_{23}+A_{32}-\omega_{23}\right)} \cdot \frac{\cos\frac{1}{2}\left(A_{41}+A_{14}+\omega_{41}\right)}{\cos\frac{1}{2}\left(A_{41}+A_{14}-\omega_{41}\right)}$$

corresponding to the stations taken in each of the three indicated orders. And in the case of any such even number n of stations (the first and last of which are coincident) it is obvious that if all the azimuths be known, and that all the differences of longitude with the exception of any two which are consecutive be known, then we can easily (by solving a quadratic equation) express the tangent of either of these two differences of longitude in terms of the known azimuths and differences of longitude.

5. With respect to any *three* mutually visible stations 1, 2, 3, we can easily arrive at convenient expressions for each of their latitudes in terms of their azimuths and differences of longitude. Thus—

We have (49) and (128)—

$$\tan\left(45^\circ - \tfrac{1}{2}\, l_1\right) \cdot \tan\left(45^\circ - \tfrac{1}{2}\, l_2\right) = - \frac{\cos \frac{1}{2}\left(A_{12} + A_{21} + \omega_{12}\right)}{\cos \frac{1}{2}\left(A_{12} + A_{21} - \omega_{12}\right)}$$

$$\frac{\tan\left(45^\circ - \frac{1}{2}\, l_1\right)}{\tan\left(45^\circ - \frac{1}{2}\, l_2\right)} = \frac{\cos \frac{1}{2}\left(A_{13} + A_{31} + \omega_{13}\right)}{\cos \frac{1}{2}\left(A_{13} + A_{31} - \omega_{13}\right)} \div \frac{\cos \frac{1}{2}\left(A_{32} + A_{23} + \omega_{32}\right)}{\cos \frac{1}{2}\left(A_{32} + A_{23} - \omega_{32}\right)}$$

$\therefore$

$$\tan^2\left(45^\circ - \tfrac{1}{2}\, l_1\right) = - \frac{\cos \frac{1}{2}\left(A_{12} + A_{21} + \omega_{12}\right)}{\cos \frac{1}{2}\left(A_{12} + A_{21} - \omega_{12}\right)} \cdot \frac{\cos \frac{1}{2}\left(A_{13} + A_{31} + \omega_{13}\right)}{\cos \frac{1}{2}\left(A_{13} + A_{31} - \omega_{13}\right)} \div \frac{\cos \frac{1}{2}\left(A_{23} + A_{32} + \omega_{23}\right)}{\cos \frac{1}{2}\left(A_{23} + A_{32} - \omega_{23}\right)}$$

$$\tan^2\left(45^\circ - \tfrac{1}{2}\, l_2\right) = - \frac{\cos \frac{1}{2}\left(A_{23} + A_{32} + \omega_{23}\right)}{\cos \frac{1}{2}\left(A_{23} + A_{32} - \omega_{23}\right)} \cdot \frac{\cos \frac{1}{2}\left(A_{21} + A_{12} + \omega_{21}\right)}{\cos \frac{1}{2}\left(A_{21} + A_{12} - \omega_{21}\right)} \div \frac{\cos \frac{1}{2}\left(A_{31} + A_{13} + \omega_{31}\right)}{\cos \frac{1}{2}\left(A_{31} + A_{13} - \omega_{31}\right)} \qquad (131)$$

$$\tan^2\left(45^\circ - \tfrac{1}{2}\, l_3\right) = - \frac{\cos \frac{1}{2}\left(A_{31} + A_{13} + \omega_{31}\right)}{\cos \frac{1}{2}\left(A_{31} + A_{13} - \omega_{31}\right)} \cdot \frac{\cos \frac{1}{2}\left(A_{32} + A_{23} + \omega_{32}\right)}{\cos \frac{1}{2}\left(A_{32} + A_{23} - \omega_{32}\right)} \div \frac{\cos \frac{1}{2}\left(A_{12} + A_{21} + \omega_{21}\right)}{\cos \frac{1}{2}\left(A_{12} + A_{21} - \omega_{21}\right)}$$

These equations are closely approximate to rigorous accuracy, even when the stations are from 100 to 200 miles asunder.

6. Let ①, ②, ③, be any three stations on the earth's spheroidal surface. Then if K_1, K_2, K_3, indicate the angles between the chords joining the stations which have their vertices in ①, ②, ③, respectively; and that C_1, C_2, C_3, indicate the corresponding angles of the geodesic triangle formed by the geodesic arcs connecting the stations; we have evidently

$$\left.\begin{aligned}
\cos C_1 &= \frac{\cos K_1}{\cos \alpha_{13} \cos \alpha_{12}} - \tan \alpha_{13} \cdot \tan \alpha_{12} \\
\cos C_2 &= \frac{\cos K_2}{\cos \alpha_{21} \cos \alpha_{23}} - \tan \alpha_{21} \cdot \tan \alpha_{23} \\
\cos C_3 &= \frac{\cos K_3}{\cos \alpha_{32} \cos \alpha_{31}} - \tan \alpha_{32} \cdot \tan \alpha_{31}
\end{aligned}\right\} \qquad (132)$$

If it were possible (and it is usually supposed so in applying LEGENDRE'S and DELAMBRE'S processes in the solution of questions pertaining to the spheroidal triangles of a trigonometrical survey) to find a sphere such that a spherical triangle described on its surface can have sides equals in length to the sides of a spheroidal triangle, and chords equal to the chords of the spheroidal triangle; then, it is obvious that by putting D_1, D_2, D_3, for the angles of this spherical triangle which correspond to the angles K_1, K_2, K_3, of the chordal triangle, we should have—

$$\left.\begin{aligned}
\cos D_1 &= \frac{\cos K_1}{\cos \frac{1}{2}(\alpha_{13} + \alpha_{31}) \cdot \cos \frac{1}{2}(\alpha_{12} + \alpha_{21}) - \tan \frac{1}{2}(\alpha_{13} + \alpha_{31}) \cdot \tan \frac{1}{2}(\alpha_{12} + \alpha_{21})} \\
\cos D_2 &= \frac{\cos K_2}{\cos \frac{1}{2}(\alpha_{21} + \alpha_{12}) \cos \frac{1}{2}(\alpha_{23} + \alpha_{32}) - \tan \frac{1}{2}(\alpha_{21} + \alpha_{12}) \cdot \tan \frac{1}{2}(\alpha_{23} + \alpha_{32})} \\
\cos D_3 &= \frac{\cos K_3}{\cos \frac{1}{2}(\alpha_{32} + \alpha_{23}) \cos \frac{1}{2}(\alpha_{31} + \alpha_{13}) - \tan \frac{1}{2}(\alpha_{32} + \alpha_{23}) \cdot \tan \frac{1}{2}(\alpha_{31} + \alpha_{13})}
\end{aligned}\right\} (133)$$

By comparing the values of the angles D_1, D_2, D_3, of the imaginary spherical triangle as given in the formulæ (133), with the correct values of the corresponding angles C_1, C_2, C_3, of the spheroidal triangle as given in formulæ (132), it is evident that, with due respect to the utmost accuracy required in practice, we have—

$$\begin{aligned}
\cos C_1 - \cos D_1 &= \tan \tfrac{1}{2}(\alpha_{13} + \alpha_{31}) \tan \tfrac{1}{2}(\alpha_{12} + \alpha_{21}) - \tan \alpha_{13} \tan \alpha_{12} \\
\cos C_2 - \cos D_2 &= \tan \tfrac{1}{2}(\alpha_{21} + \alpha_{12}) \tan \tfrac{1}{2}(\alpha_{23} + \alpha_{32}) - \tan \alpha_{21} \tan \alpha_{23} \qquad (134) \\
\cos C_3 - \cos D_3 &= \tan \tfrac{1}{2}(\alpha_{32} + \alpha_{23}) \tan \tfrac{1}{2}(\alpha_{31} + \alpha_{13}) - \tan \alpha_{32} \tan \alpha_{31}
\end{aligned}$$

their logs being the same to at least 8 or 9 places of decimals.

From these it is evident that cases may occur in geodetic surveying in which one of the angles of the spherical triangle is greater than the corresponding angle of the spheroidal triangle, and that another angle of the spherical triangle is less than its corresponding angle of the spheroidal triangle.

However the differences are very small indeed. As an instance we may consider the large spheroidal triangle of article 7, page 234, of the "Account of the Principal Triangulation of Great Britain and Ireland." Here we find that at the station whose latitude is $53°_{\prime\prime}\ 30'$, the spheroidal angle exceeds the corresponding angle of the Legendre spherical triangle by about $\frac{2}{100}$ of a second; and, although such may be disregarded in actual practice, it is nevertheless obvious that the usual method of manipulating the measured angles of a spheroidal triangle (by means of Legendre's theorem, so as to have their sum give the desired spherical excess) is erroneous in principle.

NOTES.

It is easy to perceive that the principal theorems arrived at apply to any surface whatever as well as to the surface of the spheroidal earth, even when such surface is so irregular as to be inexpressible by means of an equation.

We can assume any straight line cutting the normals to the surface at the stations S_0, S_{00}, as polar axis of reference; and then, assuming any point C_0 in this polar axis as centre of reference, we can take the plane through it perpendicular to the axis as the equatorial plane of reference. Thus the figure can be constructed as already indicated in the case in which the surface is a spheroid; and we have formulæ (50), &c.

When the stations S_0, S_{00}, are so near to each other as to permit us to regard the normals as making angles with the chord such that the ratio of their sines can be regarded as equal to unity, and the traces of the normal-chordal planes as equals in length and circular measure, we have—

$$\tan \tfrac{1}{2}\,\omega = \frac{\cos \frac{1}{2}\,(l_{\prime} - l_{\prime\prime})}{\sin \frac{1}{2}\,(l_{\prime} + l_{\prime\prime})} \cdot \cot \tfrac{1}{2}\,(A_{\prime} + A_{\prime\prime})$$

$$\tan \tfrac{1}{2}\,l' \cdot \tan \tfrac{1}{2}\,l'' = -\frac{\cos \frac{1}{2}\,(A_{\prime} + A_{\prime\prime} + \omega)}{\cos \frac{1}{2}\,(A_{\prime} + A_{\prime\prime} - \omega)}$$

$$\frac{\sin A_{\prime}}{\sin A_{\prime\prime}} = \frac{R_{\prime\prime} \cos l_{\prime\prime}}{R_{\prime} \cos l_{\prime}}$$

and all the formulæ not implicating peculiar properties of the spheroid. If there be three stations to be simultaneously considered, the assumable position for the polar axis of reference is generally restricted, as such axis must cut the three normals to the surface drawn through the stations.

If the three normals intersect in one point, any line through this point can be assumed as polar axis. If two of the normals cut each other, and that neither of them is cut by the third, then the polar axis must pass through the point of intersection and lie in the plane of this point and the third normal. If the three normals have no point of intersection, then the polar axis must lie in the surface of a ruled quadric, &c.

And when there are four stations, then should no two of the four normals lie in one plane, there can be but two transversals drawn to cut them, and therefore but two positions for the polar axis. However, with respect to all surfaces of revolution (whose normals must all cut the axis) we can arrive at general theorems applying to any stations whatever on the surface.

For instance, we can easily demonstrate the following

THEOREM.

If (1), (n), be any two stations on a surface of revolution of any kind, and $A_{1,2}$, $A_{n,n-1}$, the angles which the *true* "geodesic" joining the stations makes with the traces of the meridian planes through the stations, and that R_1, R_n, are the normals terminating in the axis, then will

$$\frac{\sin A_{1,2}}{\sin A_{n,n-1}} = \frac{R_n \cos l_n}{R_1 \cos l_1}.$$

Conceive the "geodesic" to be divided into infinitesimally small parts or elements, 1, 2; 2, 3; 3, 4; . . . . . $n-2$, $n-1$; $n-1$, n.

Let A_{12}, A_{23}, A_{34}, . . . $A_{n-1,n}$ represent the azimuths of the stations

(2), (3), (4), . . . (n) as if taken at the stations

(1), (2), (3), . . . $n-1$ respectively.

Let A_{21}, A_{32}, A_{43}, . . . $A_{n,n-1}$ represent the azimuths of the stations

(1), (2), (3), . . . $n-1$ as if taken at the stations

(2), (3), (4), . . . (n) respectively.

Let R_1, R_2, ... R_n be the lengths of the normals at stations

(1), (2), ... (n) respectively.

Then from the elements of analytic geometry, we know that the tangent lines to any infinitesimally small arc of the *first* order, which forms part of a geodesic, have their least distance apart an infinitesimally small of the *third* order; and that the ratio of the lengths of these tangents, from the points of contact to their points of least distance from each other, is that of equality. We know also that the plane of every two consecutive elements of any "geodesic" contains the normal at their point of junction; and ∴ that $\sin A_{21} = \sin A_{23}$; $\sin A_{32} = \sin A_{34}$; . . . . ; moreover, we know that the ratio of the cosines of all infinitesimally small arcs is unity. Hence we have—

$$\frac{\sin A_{12}}{\sin A_{21}} = \frac{R_2 \cos l_2}{R_1 \cos l_1}$$

$$\frac{\sin A_{23}}{\sin A_{32}} = \frac{R_3 \cos l_3}{R_2 \cos l_2}$$

$$\cdots\cdots = \cdots\cdots$$

$$\cdots\cdots = \cdots\cdots$$

And from these we at once obtain the desired proof, by equating the product of the first sides of the equations to the product of their second sides.

However, it may be proper to observe that this method of proof holds good only when none of the normals R_1, R_1, . . . R_n, is either $= 0$ or $= \infty$; and that we shall suppose this to be the case for all geodesies referred to in the present paper. We may evidently write the above relation in the form—

$$\frac{\sin A_{1,2}}{\sin A_{n,n-1}} = \frac{\text{perpendicular from } (n) \text{ to polar axis}}{\text{perpendicular from } (1) \text{ to polar axis}}$$

Or we may express it in words as follows:—

THEOREM.

On any surface of revolution, the sines of the angles $G_{\prime}$, $G_{\prime\prime}$, which the geodesic connecting two stations S_0, S_{00}, makes with the meridian traces through these stations are to each other inversely as the perpendiculars from the stations to the polar axis.

For a spheroid, such as the earth's reputed surface, we can prove, in like manner, that for any two stations whatever on its surface—

$$\frac{\sin^2 A_1}{\sin^2 A_n} = \frac{\tan^2 l_1 + \frac{a^2}{b^2}}{\tan^2 l_n + \frac{a^2}{b^2}} = \frac{\tan^2 l_1 + 1\cdot0068314987}{\tan^2 l_n + 1\cdot0068314987}$$

in which $A_{,}$, $A_{,,}$ are the angles which the true "geodesic" joining the stations makes with the meridian traces through the stations, &c.

☞ The theorem expressed by formula 10, may be expressed as follows:—

The plane perpendicular to any chord of a quadric of revolution through its middle point, bisects the portion of the axis intercepted by the normals drawn through the extremities of the chord; and the straight line joining the middle of the chord to the point in which the plane cuts the axis is divided by the equatorial plane of the surface into portions whose ratio is the same as those into which it divides either normal terminating in the axis.

From this we at once perceive that—

The perpendicular bisecting any chord of a conic bisects the portions of the axes intercepted by the normals drawn through the extremities of the chord; and that the ratio of the portions of the perpendicular measured from the middle point of the chord to its intersections with the axes, is the same as the ratio of the segments of either of the normals measured from the curve to the axes.

Problem 1.

Given the latitudes $l_{,}$ $l_{,,}$ of two stations S_{o}, S_{oo} (on the earth's spheroidal surface), and their difference of longitude ω; to find the azimuths $A_{,}$, $A_{,,}$; the circular measure Σ and length s of the geodesic arc between the stations; the angles $a_{,}$ $a_{,,}$ of depression of the chord, &c.

First Method.

To find the arcs L', L'', and the azimuths $A_{,}$ $A_{,,}$ we have—

$$\cot L' = e^2 \cdot \frac{R_{,,} \sin l_{,,}}{R_{,} \cos l_{,}} + (1 - e^2) \tan l_{,}$$

$$\cot L'' = e^2 \cdot \frac{R_{,} \sin l_{,}}{R_{,,} \cos l_{,,}} + (1 - e^2) \tan l_{,,}$$

$$\cot A_{,} = \frac{\cot L'' \cos l_{,} - \sin l_{,} \cos \omega}{\sin \omega}$$

$$\cot A_{,,} = \frac{\cot L' \cos l_{,,} - \sin l_{,,} \cos \omega}{\sin \omega}$$

or having found the arcs L', L'', as above indicated, we can find the azimuths and the angles $D_,$, $D_{,,}$ by means of the formulæ—

$$\tan \tfrac{1}{2} (A_, + D_{,,}) = \frac{\cos \frac{1}{2} (L'' - l')}{\cos \frac{1}{2} (L'' + l')} \cdot \cot \tfrac{1}{2} \omega$$

$$\tan \tfrac{1}{2} (A_, - D_{,,}) = \frac{\sin \frac{1}{2} (L'' - l')}{\sin \frac{1}{2} (L'' + l')} \cdot \cot \tfrac{1}{2} \omega$$

$$\tan \tfrac{1}{2} (D_, + A_{,,}) = \frac{\cos \frac{1}{2} (l'' - L')}{\cos \frac{1}{2} (l'' + L')} \cdot \cot \tfrac{1}{2} \omega$$

$$\tan \tfrac{1}{2} (D_, - A_{,,}) = \frac{\sin \frac{1}{2} (l'' - L')}{\sin \frac{1}{2} (l'' + L')} \cdot \cot \tfrac{1}{2} \omega$$

To find $a_,$, $a_{,,}$, Σ, $z_,$, $z_{,,}$ and s, we may proceed as follows:—
First we find $\delta_,$, $\delta_{,,}$ from

$$\delta_, = L' - l'$$

$$\delta_{,,} = l'' - L''$$

Then from the triangles $S_, ID_,$, $S_{,,} ID_{,,}$, we have, to find $IS_,$, $ID_,$, $IS_{,,}$, $ID_{,,}$—

$$\tan \tfrac{1}{2} (IS_, + ID_,) = \frac{\sin \frac{1}{2} (D_, + A_,)}{\sin \frac{1}{2} (D_, - A_,)} \cdot \tan \tfrac{1}{2} \delta_,$$

$$\tan \tfrac{1}{2} (IS_, - ID_,) = \frac{\cos \frac{1}{2} (D_, + A_,)}{\cos \frac{1}{2} (D_, - A_,)} \cdot \tan \tfrac{1}{2} \delta_,$$

$$\tan \tfrac{1}{2} (IS_{,,} + ID_{,,}) = \frac{\sin \frac{1}{2} (A_{,,} + D_{,,})}{\sin \frac{1}{2} (A_{,,} - D_{,,})} \cdot \tan \tfrac{1}{2} \delta_{,,}$$

$$\tan \tfrac{1}{2} (IS_{,,} - ID_{,,}) = \frac{\cos \frac{1}{2} (A_{,,} + D_{,,})}{\cos \frac{1}{2} (A_{,,} - D_{,,})} \cdot \tan \tfrac{1}{2} \delta_{,,}$$

Then—

$$a_, = 90° - IS_,$$

$$a_{,,} = IS_{,,} - 90°$$

$$\Sigma = a_, + a_{,,}$$

$$z_, = ID_{,,} - IS_,$$

$$z_{,,} = IS_{,,} - ID_,$$

$$s = z_, \cdot R_, \cdot \sin 1'' = z_{,,} \cdot R_{,,} \cdot \sin 1''$$

But we can find k and s otherwise, thus—

$$k = \frac{R_, \cos l_, \sin \omega}{\sin A_{,,} \cos a_{,,}} = \frac{R_{,,} \cos l_{,,} \sin \omega}{\sin A_, \cos a_,}$$

$$s = k \cdot \frac{\Sigma \cdot \sin 1''}{2 \cdot \sin \frac{1}{2} \Sigma}$$

Or having found k, in terms of the given data, from

$$k^2 = (R_, \cos l_,)^2 + (R_{,,} \cos l_{,,})^2 - 2 \cdot R_, \cdot R_{,,} \cos l_, \cos l_{,,} \cos \omega + (1 - e^2)^2 \cdot (R_, \sin l_, - R_{,,} \sin l_{,,})^2$$

we can find the angles of depression $a_{,}$, $a_{,,}$ by means of (109), and then find the azimuths from

$$\sin A_{,} = \frac{R_{,,} \cos l_{,,} \cos \omega}{k \cdot \cos a_{,}}$$

$$\sin A_{,,} = \frac{R_{,} \cos l_{,} \cos \omega}{k \cdot \cos a_{,,}}$$

☞ When $A_{,}$ or $A_{,,}$ is found to be nearly 90°, it cannot be accurately obtained by means of the usual tables of logarithms; so that, in such case, it is necessary to proceed as indicated in the works on trigonometry. Thus, putting A for the angle to be found, and N for the value of the function to which sin A is equated (which is nearly equal to 1), we have—

$$\sin (45° - \tfrac{1}{2} A) = \sqrt{\frac{1 - N}{2}}$$

or,

$$\tan (45° - \tfrac{1}{2} A) = \sqrt{\frac{1 - N}{1 + N}}$$

from which to compute the value of the angle A.

And when, in the sequel, an angle is to be found from an expression for its sine which is nearly equal to unity; then, putting N to represent such expression, we should proceed to find the angle by these formulæ.

Otherwise.

(When the stations are not more than 40 miles asunder.)

From the spherical triangle $S_{,}PS_{,,}$ we have the formulæ—

$$\tan \tfrac{1}{2} (A_{\circ} + A_{\circ\circ}) = \frac{\cos \tfrac{1}{2} (l'' - l')}{\cos \tfrac{1}{2} (l' + l'')} \cdot \cot \tfrac{1}{2} \omega$$

$$\tan \tfrac{1}{2} (A_{\circ} - A_{\circ\circ}) = \frac{\sin \tfrac{1}{2} (l'' - l')}{\sin \tfrac{1}{2} (l'' + l')} \cdot \cot \tfrac{1}{2} \omega$$

$$\sin \nu = \frac{\sin l' \sin \omega}{\sin A_{\circ\circ}} = \frac{\sin l'' \sin \omega}{\sin A_{\circ}}$$

Then to find the azimuths we have—

$$\tan x = \frac{R_{,,} \sin l''}{R_{,} \sin l'}$$

$$\tan \tfrac{1}{2} (A_{,} - A_{,,}) = \tan \tfrac{1}{2} (A_{\circ} + A_{\circ\circ}) \tan (x - 45°)$$

$$\tfrac{1}{2} (A_{,} + A_{,,}) = \tfrac{1}{2} (A_{\circ} + A_{\circ\circ})$$

To find Ω, Σ, and the angle $\triangle$, we have—

$$\Omega = \text{A}_{\circ} - \text{A}_{,} = \text{A}_{,,} - \text{A}_{\circ\circ}$$

$$\tan \tfrac{1}{2} \Sigma = \tan \tfrac{1}{2} \nu \cos \Omega, \text{ or } \Sigma = \nu \cdot \cos \Omega$$

$$\triangle = 2 \cdot \Omega \cdot \sin \tfrac{1}{2} \Sigma, \text{ or } \triangle = \Omega \cdot \Sigma \cdot \sin 1''$$

To find the length k of the geodesic chord between the stations—

$$k = \frac{\text{R}_{,} \sin l' \sin \omega}{\sin \text{A}_{,,} \cos \frac{1}{2} \Sigma} = \frac{\text{R}_{,,} \sin l'' \sin \omega}{\sin \text{A}_{,} \cos \frac{1}{2} \Sigma}$$

Then to find s, we have—

$$s = \frac{k \cdot \Sigma'' \cdot \sin 1''}{2 \cdot \sin \frac{1}{2} \Sigma}$$

And to find the angles $a_{,,}$, $a_{,}$, of depression of the chord k below the tangent planes to the earth at the stations $\text{S}_{\circ\circ}$, $\text{S}_{\circ}$, we have—

$$\tan y = \frac{\text{R}_{,}}{\text{R}_{,,}}$$

$$(a_{,,} - a_{,}) = (y - 45^{\circ}) \cdot \Sigma \cdot \sin 1''$$

$$(a_{,,} + a_{,}) = \Sigma.$$

Problem 2.

Given the latitude $l_{,}$, the azimuth $\text{A}_{,}$, and the length s and circular measure Σ of the geodesic arc between the stations; to find the latitude $l_{,,}$, the azimuth $\text{A}_{,,}$, the difference of longitude ω, &c.

First Method.

To find the angle $\phi_{,}$, we have, from the spherical triangle $\text{PS}_{,}\text{I}$—

$$\tan \tfrac{1}{2} (\phi_{,} + \beta_{,}) = \frac{\cos \frac{1}{2} (l_{,} - \frac{1}{2} \Sigma)}{\sin \frac{1}{2} (l_{,} + \frac{1}{2} \Sigma)} \cdot \tan \tfrac{1}{2} \text{A}_{,}$$

$$\tan \tfrac{1}{2} (\phi_{,} - \beta_{,}) = \frac{\sin \frac{1}{2} (l_{,} - \frac{1}{2} \Sigma)}{\cos \frac{1}{2} (l_{,} + \frac{1}{2} \Sigma)} \cdot \tan \tfrac{1}{2} \text{A}_{,}$$

☞ It may be proper to observe that $\frac{1}{2} \Sigma$ is used in these formulas instead of the angle $a_{,}$ of depression of the chord; but as the difference of these will in all actual cases be less than $\frac{1}{10}$ of a second, and that the numerators vary as the denominators when $\frac{1}{2} \Sigma$ varies in value, and that any variation in $\frac{1}{2} \Sigma$ which increases or decreases $\frac{1}{2} (\phi_{,} + \beta_{,})$ will decrease or increase $\frac{1}{2} (\phi_{,} - \beta_{,})$; ∴, as respects the value of

$\phi_, = \frac{1}{2}(\phi_{,,} + \beta_,) + \frac{1}{2}(\phi_, - \beta_,)$, there can be no appreciable difference whether we use $\frac{1}{2}\Sigma$ or $a_,$.

Find the chord k by means of the usual formula—

$$k = \frac{2 \cdot s \cdot \sin \frac{1}{2} \Sigma}{\Sigma \cdot \sin 1''}.$$

Then, to find the difference of longitude ω, and the angle $\phi_{,,}$ by means of the plane triangle $p_, C_, p_{,,}$, we have—

$$\tan h_, = R_, \cos l_, ; \quad \tan h_{,,} = \frac{k \cdot \sin A_, \cos \frac{1}{2} \Sigma}{\sin \phi_,}$$

$$\frac{1}{2}(\phi_{,,} + \omega) = 90° - \frac{1}{2}\phi_,$$

$$\tan \frac{1}{2}(\phi_{,,} - \omega) = \frac{\sin (h_{,,} - h_,)}{\sin (h_{,,} + h_,)} \cdot \cot \frac{1}{2}\phi_,$$

Then to find the azimuth $A_{,,}$ and latitude $l_{,,}$, we have—

$$\sin A_{,,} = \frac{\sin \phi_{,,}}{\sin \phi_,} \cdot \sin A_,$$

$$\tan \frac{1}{2} l'' = -\frac{\cos \frac{1}{2}(A_, + A_{,,} + \omega)}{\cos \frac{1}{2}(A_, + A_{,,} - \omega)} \cdot \cot \frac{1}{2} l'$$

☞ If instead of $l_,$ $A_,$ we were given $l_{,,}$, $A_{,,}$, we should first proceed to find the angle $\phi_{,,}$ by means of—

$$\tan \frac{1}{2}(\phi_{,,} + \beta_{,,}) = \frac{\cos \frac{1}{2}(l_{,,} - \frac{1}{2}\Sigma)}{\sin \frac{1}{2}(l_{,,} + \frac{1}{2}\Sigma)} \cdot \tan \frac{1}{2} A_{,,}$$

$$\tan \frac{1}{2}(\phi_{,,} - \beta_{,,}) = \frac{\sin \frac{1}{2}(l_{,,} - \frac{1}{2}\Sigma)}{\cos \frac{1}{2}(l_{,,} + \frac{1}{2}\Sigma)} \cdot \tan \frac{1}{2} A_{,,}$$

and then proceed in an analogous manner to find $\phi_,$, ω, $A_,$, and $l_{,,}$.

Otherwise (Case 1st).

Given $l_,$ $A_,$ s; to find ω, $l_{,,}$, and $A_{,,}$ (see foot-note).

To find $z_,$ $D_{,,}$, ω, and L'', we have—

$$z_, = \frac{s}{R_, \sin 1''}$$

$$\tan \frac{1}{2}(D_{,,} + \omega) = \frac{\cos \frac{1}{2}(l' - z_,)}{\cos \frac{1}{2}(l' + z_,)} \cdot \cot \frac{1}{2} A_,$$

$$\tan \frac{1}{2}(D_{,,} - \omega) = \frac{\sin \frac{1}{2}(l' - z_,)}{\sin \frac{1}{2}(l' + z_,)} \cdot \cot \frac{1}{2} A_,$$

$$\tan \frac{1}{2}(L'' - l') = \frac{\sin \frac{1}{2}(A_, - D_{,,})}{\sin \frac{1}{2}(A_, + D_{,,})} \cdot \tan \frac{1}{2} z_,$$

or,

$$\sin L'' = \frac{\sin l' \sin A_,}{\sin D_{,,}}.$$

Then to find $\delta_{,,}$, $l_{,,}$, and $A_{,,}$ we have—

$$\delta_{,,} = \left(\frac{e^2}{1 - e^2}\right) \cdot \sin L'' \sin \tfrac{1}{2} (L'' + l') \cdot (L'' - l')$$

$$l_{,,} = 90° - (L'' + \delta_{,,})$$

$$A_{,,} - D_{,,} = \sin D_{,,} \cdot \tan \tfrac{1}{2} z_{,} \cdot \delta_{,,}$$

☞ This case, in which the given latitude $l_{,}$ is greater than the sought latitude $l_{,,}$, is made known to us by the given azimuth $A_{,}$ being greater than the computed angle $D_{,,}$. And as we must have (see formulæ 21) the sought azimuth $A_{,,}$ also greater than the angle $D_{,,}$ it is evident that by putting ζ to represent the excess, we have—

$$\tan \tfrac{1}{2} (A_{,,} + \omega - \zeta) = \frac{\cos \tfrac{1}{2} (l' - z_{,})}{\cos \tfrac{1}{2} (l' + z_{,})} \cdot \cot \tfrac{1}{2} A_{,}$$

$$\tan \tfrac{1}{2} (A_{,,} - \omega - \zeta) = \frac{\sin \tfrac{1}{2} (l' - z_{,})}{\sin \tfrac{1}{2} (l' + z_{,})} \cdot \cot \tfrac{1}{2} A_{,}$$

shewing that the formulæ given in the "Account of the Principal Triangulation of Great Britain and Ireland" (see pages 247, 249, 676 of that work) are erroneous in every case in which the given latitude is greater than the sought latitude.

(Case 2nd.)

Given $l_{,,}$, $A_{,,}$, s; to find ω, $l_{,}$ and $A_{,}$.

To find $z_{,,}$, $D_{,}$, ω, L', we have—

$$z_{,,} = \frac{s}{R_{,,} \cdot \sin 1''}$$

$$\tan \tfrac{1}{2} (D_{,} + \omega) = \frac{\cos \tfrac{1}{2} (l'' - z_{,,})}{\cos \tfrac{1}{2} (l'' + z_{,,})} \cdot \cot \tfrac{1}{2} A_{,,}$$

$$\tan \tfrac{1}{2} (D_{,} - \omega) = \frac{\sin \tfrac{1}{2} (l'' - z_{,,})}{\sin \tfrac{1}{2} (l'' + z_{,,})} \cdot \cot \tfrac{1}{2} A_{,,}$$

$$\tan \tfrac{1}{2} (l'' - L') = \frac{\sin \tfrac{1}{2} (D_{,} - A_{,,})}{\sin \tfrac{1}{2} (D_{,} + A_{,,})} \cdot \tan \tfrac{1}{2} z_{,,}$$

or,

$$\sin L' = \frac{\sin l'' \cdot \sin A_{,,}}{\sin D_{,}}$$

To find $\delta_{,}$, $l_{,}$ and $A_{,}$, we have—

$$\delta_{,} = \left(\frac{e^2}{1 - e^2}\right) \cdot \sin L' \cdot \sin \tfrac{1}{2} (l'' + L') \cdot (l'' - L')$$

$$l_{,} = 90° - (L' - \delta_{,})$$

$$D_{,} - A_{,} = \sin D_{,} \cdot \tan \tfrac{1}{2} z_{,,} \cdot \delta_{,}$$

☞ This case, in which the given or known latitude $l_{\prime\prime}$ is less than the sought latitude $l_{\prime}$, will be intimated to us by the angles $A_{\prime\prime}$ and $D_{\prime}$; we shall have the given azimuth $A_{\prime\prime}$ less than the angle $D_{\prime}$. If the angle $A_{\prime\prime} = D_{\prime}$, then $A_{\prime} = D_{\prime}$, and $l_{\prime} = l_{\prime\prime}$, &c.

Otherwise.

Case 1°. When $l_{\prime}$, $A_{\prime}$, s, are given; to find $l_{\prime\prime}$, $A_{\prime\prime}$, ω.

Find $z_{\prime}$, ω, $D_{\prime\prime}$, as indicated in the last solution, and then find $A_{\prime\prime}$ by means of—

$$\sin A_{\prime\prime} = \frac{\cos (z_{\prime} - \frac{1}{2}\Sigma)}{\cos \frac{1}{2}\Sigma} \cdot \sin D_{\prime\prime}$$

And find $l_{\prime\prime}$ from—

$$\tan \tfrac{1}{2} l'' = -\frac{\cos \frac{1}{2}(A_{\prime} + A_{\prime\prime} + \omega)}{\cos \frac{1}{2}(A_{\prime} + A_{\prime\prime} - \omega)} \cdot \cot \tfrac{1}{2} l'$$

$$l_{\prime\prime} = 90^\circ - l''.$$

Case 2°. When $l_{\prime\prime}$, $A_{\prime\prime}$, s, are given; to find $l_{\prime}$, $A_{\prime}$, ω.

Find $z_{\prime\prime}$, ω, $D_{\prime}$, as indicated in the last solution, and then find $A_{\prime}$ by means of—

$$\sin A_{\prime} = \frac{\cos (z_{\prime\prime} - \frac{1}{2}\Sigma)}{\cos \frac{1}{2}\Sigma} \cdot \sin D_{\prime}$$

And find $l_{\prime}$ from—

$$\tan \tfrac{1}{2} l' = -\frac{\cos \frac{1}{2}(A_{\prime} + A_{\prime\prime} + \omega)}{\cos \frac{1}{2}(A_{\prime} + A_{\prime\prime} - \omega)} \cdot \cot \tfrac{1}{2} l''$$

$$l_{\prime} = 90^\circ - l'.$$

Problem 3.

Given the latitudes $l_{\prime}$, $l_{\prime\prime}$, and the azimuth $A_{\prime}$; to find the azimuth $A_{\prime\prime}$, the difference of longitude ω, &c.

By equating the values of $\sin \alpha$, as expressed in formulæ 108, 109, we have—

$$R_{\prime\prime} \cos l_{\prime\prime} (\cos^2 l_{\prime} + 1) \sqrt{1 - \sin^2 \omega}$$
$$= (R_{\prime} + \frac{a^2}{R_{\prime}} - R_{\prime\prime} \cdot \frac{b^2}{a^2} \cdot \sin l_{\prime} \sin l_{\prime\prime}) \cos l_{\prime}$$
$$- (R_{\prime\prime} \cos l_{\prime\prime} \tan l_{\prime} \cot A_{\prime}) \sin \omega$$

or, $$M \cdot \sqrt{1 - \sin^2 \omega} = L - N \cdot \sin \omega$$

in which the values of M, L, and N are known.

From this we at once obtain

$$\sin \omega = \frac{L N + \sqrt{M^2 (M^2 + N^2 - L^2)}}{M^2 + N^2}$$

in which the + sign only should precede the radical portion. This is evident. For since the general expression for sin ω holds when $A_, = 90°$, in which case $N = 0$; and that sin ω must be positive; therefore it is the + sign that must in such case, and in all cases, precede the radical.

We may also find ω in the following manner—

Find the arc L'' by means of formula (79), and the angle $D_{,,}$ from—

$$\sin D_{,,} = \frac{\cos l_, \sin A_,}{\sin L''}$$

and then to find ω we have—

$$\tan \tfrac{1}{2} \omega = \frac{\cos \frac{1}{2} (L'' - l')}{\cos \frac{1}{2} (L'' + l')} \cdot \cot \tfrac{1}{2} (A_, + D_{,,})$$

To find the azimuth $A_{,,}$ we then have—

$$\tan \tfrac{1}{2} (A_, + A_{,,}) = \frac{\cos \frac{1}{2} (l_, - l_{,,})}{\sin \frac{1}{2} (l_, + l_{,,})} \cdot \cot \tfrac{1}{2} \omega$$

And to find s, we have—

$$\sin z_, = \frac{\sin L'' \sin \omega}{\sin A_,}$$

$$s = z_, \cdot R_, \cdot \sin 1''$$

The other entities can be easily found as indicated by formulæ.

☞ If $l_{,,}$, $l_,$, $A_{,,}$ were given instead of $l_,$, $l_{,,}$, $A_,$; then instead of L'', $D_{,,}$, &c., in the preceding formulæ, we should have L', $D_,$, &c.

Otherwise.

To find the azimuth $A_{,,}$, we have—

$$\sin A_{,,} = \frac{R_, \cdot \cos l_,}{R_{,,} \cdot \cos l_{,,}} \cdot \sin A_, \text{ nearly.}$$

And then to find ω, we have—

$$\tan \tfrac{1}{2} \omega = \frac{\cos \frac{1}{2} (l_, - l_{,,})}{\sin \frac{1}{2} (l_, + l_{,,})} \cdot \cot \tfrac{1}{2} (A_, + A_{,,})$$

And when instead of $A_,$ the azimuth $A_{,,}$ is given, the first of these must be replaced by

$$\sin A_, = \frac{R_{,,} \cdot \cos l_{,,}}{R_, \cdot \cos l_,} \cdot \sin A_{,,}$$

&c., &c.

PROBLEM 4.

Given the two azimuths $A_{,}$, $A_{,,}$, and one of the latitudes $l_{,}$; to find the latitude $l_{,,}$, the difference of longitude ω of the stations, &c.

To find the latitude $l_{,,}$, we have, from (53)—

$$\tan^2 l_{,,} = \frac{(1 - e^2) \tan^2 l_{,} \sin^2 A_{,,} - (\sin^2 A_{,} - \sin^2 A_{,,})}{(1 - e^2) \sin^2 A_{,}} \text{ nearly.}$$

Then to find the difference of longitude, we have—

$$\tan \tfrac{1}{2} \omega = \frac{\cos \frac{1}{2} (l_{,} - l_{,,})}{\sin \frac{1}{2} (l_{,} + l_{,,})} \cdot \cot \tfrac{1}{2} (A_{,} + A_{,,})$$

The other entities can now be found, &c.

PROBLEM 5.

Given the latitude $l_{,}$, the azimuth $A_{,}$, and the difference of longitude ω; to find the latitude $l_{,,}$, the azimuth $A_{,,}$, &c.

Find L″ by means of formula 78.

Then finding m, p, q, by means of—

$$m = \cot^2 L'' - \frac{e^4}{a^2} \cdot R^2_{,} \cdot \sin^2 l_{,}$$

$$p = \cot^2 L'' - \frac{e^6}{a^2} \cdot R^2 \cdot \sin^2 l_{,} + (1 - e^2)^2$$

$$q = 2\, e^2 (1 - e^2) \frac{R_{,}}{a} \cdot \sin l_{,}$$

the second of the formulæ 79, gives us the equation—

$$m - p \cdot \sin^2 l_{,,} = q \cdot \sin l_{,,} \sqrt{1 - e^2 \cdot \sin^2 l_{,,}}$$

And from this we immediately obtain—

$$\sin^2 l_{,,} = \frac{q^2 + 2\, m\, p + q \sqrt{q^2 + 4\, m\, (p - m\, e^2)}}{2\, (p^2 + q^2\, e^2)}$$

Now, if we conceive a case in which $l_{,}$ is of any value we wish, and that the corresponding value of $l_{,,}$ is such that $m = 0$; then it is evident $l_{,,}$, p, q, have finite values; and we perceive that in such case the + sign only must precede the radical. And it is ∴ evident that the + sign must, in all cases, precede the radical in the above general expression for $\sin^2 l_{,,}$.

Or we may proceed as follows—

From the triangle $S_{,}PD_{,,}$, we have to find L″, $z_{,}$, $D_{,,}$

$$\tan \tfrac{1}{2} (L'' + z_{,}) = \frac{\cos \frac{1}{2} (A_{,} - \omega)}{\cos \frac{1}{2} (A_{,} + \omega)} \cdot \tan \tfrac{1}{2} l'$$

$$\tan \tfrac{1}{2} (L'' - z_{,}) = \frac{\sin \frac{1}{2} (A_{,} - \omega)}{\sin \frac{1}{2} (A_{,} + \omega)} \cdot \tan \tfrac{1}{2} l'$$

$$\sin D_{,,} = \frac{\sin l' \cdot \sin A_{,}}{\sin L''} = \frac{\sin l' \cdot \sin \omega}{\sin z_{,}}$$

or,

$$\tan \tfrac{1}{2} (A_{,} - D_{,,}) = \frac{\sin \frac{1}{2} (L'' - l')}{\sin \frac{1}{2} (L'' + l')} \cdot \cot \tfrac{1}{2} \omega$$

Then we can find $\delta_{,,}$ by 83 or any of the formulæ 88, and the azimuth $A_{,,}$ by means of any of the formulæ 94.

Then, $l_{,,} = 90° - (L'' + \delta_{,,})$. &c., &c.

When instead of $l_{,}$, $A_{,}$, we are given $l_{,,,}$, $A_{,,,}$, the analogous methods of proceeding are evident.

Problem 6.

Given the azimuth $A_{,,}$ the latitude $l_{,,}$, and the length s and circular measure Σ of the arc between the stations; to find $A_{,,,}$, $l_{,}$, ω, &c.

To find ω, $z_{,,}$, $D_{,}$, $A_{,,,}$ and $l_{,}$ we have—

$$\sin \omega = \frac{s \cdot \sin \Sigma \cdot \sin A_{,}}{R_{,,} \cdot \Sigma \cdot \cos l_{,,} \cdot \sin 1''}$$

$$z_{,,} = \frac{s}{R_{,,} \cdot \sin 1''}$$

$$\sin D_{,} = \frac{\cos l_{,,} \cdot \sin \omega}{\sin z_{,,}}$$

$$\tan \tfrac{1}{2} A_{,,} = \frac{\sin \frac{1}{2} (l'' - z_{,,})}{\sin \frac{1}{2} (l'' + z_{,,})} \cdot \cot \tfrac{1}{2} (D_{,} - \omega)$$

$$\tan \tfrac{1}{2} l' = - \frac{\cos \frac{1}{2} (A_{,} + A_{,,} + \omega)}{\cos \frac{1}{2} (A_{,} + A_{,,} - \omega)} \cdot \cot \tfrac{1}{2} l''$$

If $A_{,,,}$, $l_{,}$ were given instead of $A_{,,}$ $l_{,,}$, the method of solution is analogous, and requires no particular elucidation.

Problem 7.

Given the latitude $l_{,}$, the difference of longitude ω, and the length s and circular measure Σ of the arc between the stations; to find the azimuths $A_{,}$, $A_{,,}$, the latitude $l_{,,}$, &c.

To find $z_{,}$, $D_{,,}$, $A_{,}$, $A_{,,}$, $l_{,,}$ we have—

$$z_{,} = \frac{s}{R_{,} \cdot \sin 1''}.$$

$$\sin D_{\prime\prime} = \frac{\sin l' \sin \omega}{\sin z_{\prime}}$$

$$\tan \tfrac{1}{2} A_{\prime} = \frac{\sin \frac{1}{2} (l' - z_{\prime})}{\sin \frac{1}{2} (l' + z_{\prime})} \cdot \cot \tfrac{1}{2} (D_{\prime\prime} - \omega)$$

$$\sin A_{\prime\prime} = \frac{R_{\prime} \cdot \Sigma \cdot \cos l_{\prime} \sin \omega}{s \cdot \sin \Sigma}$$

$$\tan \tfrac{1}{2} l'' = -\frac{\cos \frac{1}{2} (A_{\prime} + A_{\prime\prime} + \omega)}{\cos \frac{1}{2} (A_{\prime} + A_{\prime\prime} - \omega)} \cdot \cot \tfrac{1}{2} l'$$

And similarly when $l_{\prime\prime}$ is given instead of $l_{\prime}$.

Problem 8.

Given the azimuth $A_{\prime}$, the difference of longitude ω, and the length s and circular measure Σ of the arc between the stations; to find the latitudes, &c.

Putting—
$$G = \frac{s \cdot \sin \Sigma'' \cdot \sin A}{\sin \omega \cdot \Sigma'' \cdot \sin 1''}$$

We easily find, from 62—

$$\sin l_{\prime\prime} = \sqrt{\frac{(a + G) \cdot (a - G)}{(a + eG) \cdot (a - eG)}}$$

And now we can find the other entities as in problems 6 and 7.

Problem 9.

Given the two latitudes $l_{\prime}$, $l_{\prime\prime}$, and the length s and circular measure Σ of the arc between the stations; to find the azimuths $A_{\prime}$, $A_{\prime\prime}$, &c.

To find L', L'', $z_{\prime}$, $z_{\prime\prime}$, we have—

$$\cot L' = e^2 \cdot \frac{R_{\prime\prime} \sin l_{\prime\prime}}{R_{\prime} \cos l_{\prime}} + (1 - e^2) \tan l_{\prime}$$

$$\cot L'' = e^2 \cdot \frac{R_{\prime} \sin l_{\prime}}{R_{\prime\prime} \cos l_{\prime\prime}} + (1 - e^2) \tan l_{\prime\prime}$$

$$z_{\prime} = \frac{s}{R_{\prime} \cdot \sin 1''}$$

$$z_{\prime\prime} = \frac{s}{R_{\prime\prime} \cdot \sin 1''}$$

Then from the spherical triangles $S_{\prime}PD_{\prime\prime}$, $S_{\prime\prime}PD_{\prime}$, we have —putting $p = \frac{1}{2} (l' + z_{\prime} + L'')$, $q = \frac{1}{2} (l'' + z_{\prime\prime} + L')$,—

$$\tan^2 \tfrac{1}{2} A_{\prime} = \frac{\sin (p - z_{\prime}) \sin (p - l')}{\sin p \sin (p - L'')}$$

$$\tan^2 \tfrac{1}{2} A_{\prime\prime} = \frac{\sin (q - z_{\prime\prime}) \sin (q - l'')}{\sin q \sin (q - L')}$$

$$\tan^2 \tfrac{1}{2} \omega = \frac{\sin (p - L'') \sin (p - l')}{\sin p \sin (p - z_{\prime})}$$

$$\tan^2 \tfrac{1}{2} \omega = \frac{\sin (q - L') \sin (q - l'')}{\sin q \sin (q - z_{\prime\prime})}$$

In this method of solution we have not made use of Σ. In the following method we shall not make use of *s*, but of Σ; and it is applicable to any two stations on the earth's spheroidal surface, as well as to mutually visible stations.

Otherwise.

Find the angles $a_{\prime\prime}$, $a_{\prime}$, of depression of the chord by means of—

$$\tan x = \frac{R_{\prime}}{R_{\prime\prime}}$$

$$\tan \tfrac{1}{2} (\alpha_{\prime\prime} - \alpha_{\prime}) = \tan (x - 45°) \cdot \tan \tfrac{1}{2} \Sigma$$

$$\tfrac{1}{2} (\alpha_{\prime\prime} + \alpha_{\prime}) = \tfrac{1}{2} \Sigma$$

To find the azimuths we have the equations—

$$\cos \alpha_{\prime} \cos l_{\prime} \cos A_{\prime} + \cos \alpha_{\prime\prime} \cos l_{\prime\prime} \cos A_{\prime\prime} = \sin \alpha_{\prime} \sin l_{\prime} + \sin \alpha_{\prime\prime} \sin l_{\prime\prime}$$

$$\frac{1 - \cos^2 A_{\prime}}{1 - \cos^2 A_{\prime\prime}} = \frac{(R_{\prime\prime} \cos \alpha_{\prime\prime} \cos l_{\prime\prime})^2}{(R_{\prime} \cos \alpha_{\prime} \cos l_{\prime})^2}$$

By putting

$M_{\prime} = \cos \alpha_{\prime} \cos l_{\prime}$; $M_{\prime\prime} = \cos \alpha_{\prime\prime} \cos l_{\prime\prime}$; $Q = \sin \alpha_{\prime} \sin l_{\prime} + \sin \alpha_{\prime\prime} \sin l_{\prime\prime}$

we easily find—

$$\cos A_{\prime} = \frac{-Q R^2_{\prime\prime} + \sqrt{(Q \cdot R_{\prime} \cdot R_{\prime\prime})^2 - (R^2_{\prime} - R^2_{\prime\prime}) \cdot (M^2_{\prime\prime} \cdot R^2_{\prime\prime} - M^2_{\prime} \cdot R^2_{\prime})}}{M_{\prime} \cdot (R^2_{\prime} - R^2_{\prime\prime})}$$

$$\cos A_{\prime\prime} = \frac{Q \cdot R^2_{\prime} - \sqrt{(Q \cdot R_{\prime} \cdot R_{\prime\prime})^2 - (R^2_{\prime} - R^2_{\prime\prime}) \cdot (M^2_{\prime\prime} \cdot R^2_{\prime\prime} - M^2_{\prime} \cdot R^2_{\prime})}}{M_{\prime\prime} \cdot (R^2_{\prime} - R^2_{\prime\prime})}$$

Since $\cos A_{\prime}$ must be positive when the angle $A_{\prime}$ is acute, ∴ it is evident that in all cases it is the + sign which must precede the radical in the above expression for $\cos A_{\prime}$. It is evident that in the expression for $\cos A_{\prime\prime}$, it is the — sign only which should precede the radical.

☞ When $l_{\prime} = l_{\prime\prime}$; then $a_{\prime\prime} = a_{\prime}$; $R_{\prime} = R_{\prime\prime}$; $M_{\prime} = M_{\prime\prime}$; and the above expressions can be written in the forms—

$$\cos A_{\prime} = \frac{Q R_{\prime\prime} (R_{\prime} - R_{\prime\prime})}{M_{\prime} (R_{\prime} + R_{\prime\prime}).(R_{\prime} - R_{\prime\prime})}$$

$$\cos A_{\prime\prime} = \frac{Q\, R_{\prime}\, (R_{\prime} - R_{\prime\prime})}{M_{\prime\prime}\, (R_{\prime} + R_{\prime\prime})\, (R_{\prime} - R_{\prime\prime})}$$

$$\therefore\ \cos A_{\prime} = \cos A_{\prime\prime} = \frac{Q}{2\,M} = \tan \tfrac{1}{2}\,\Sigma \cdot \tan l_{\prime}.$$

Otherwise.

To find the chord k and the angle θ which it makes with the polar axis, we have—

$$k = \frac{2\,s \cdot \sin \frac{1}{2}\,\Sigma}{\Sigma}$$

$$\cos \theta = \frac{1 - e^2}{k} \cdot (R_{\prime} \sin l_{\prime} - R_{\prime\prime} \sin l_{\prime\prime})$$

To find the sides of the plane triangle $p_{\prime}\, C_{\circ}\, p_{\prime\prime}$, we have—

$$C_{\circ}\, p_{\prime} = R_{\prime} \cos l_{\prime};\ C\, p_{\prime\prime} = R_{\prime\prime} \cos l_{\prime\prime};\ p_{\prime} p_{\prime\prime} = k \cdot \sin \theta.$$

And knowing the three sides of this plane triangle, we can find its angles $\phi_{\prime}$, $\phi_{\prime\prime}$, ω.

Then from the spherical triangles $S_{\prime}PI$, $S_{\prime\prime}PI$, we have the following formulæ from which to obtain the azimuths—

$$\cot \tfrac{1}{2}\,(A_{\prime} - \psi) = \frac{\cos \frac{1}{2}\,(\theta - l')}{\cos \frac{1}{2}\,(\theta + l')} \cdot \cot \tfrac{1}{2}\,\phi_{\prime};$$

$$\tan \tfrac{1}{2}\,(A_{\prime\prime} + \psi) = \frac{\cos \frac{1}{2}\,(\theta - l'')}{\cos \frac{1}{2}\,(\theta + l'')} \cdot \tan \tfrac{1}{2}\,\phi_{\prime\prime}$$

$$\cot \tfrac{1}{2}\,(A_{\prime} + \psi) = \frac{\sin \frac{1}{2}\,(\theta - l')}{\sin \frac{1}{2}\,(\theta + l')} \cdot \cot \tfrac{1}{2}\,\phi_{\prime};$$

$$\tan \tfrac{1}{2}\,(A_{\prime\prime} - \psi) = \frac{\sin \frac{1}{2}\,(\theta - l'')}{\sin \frac{1}{2}\,(\theta + l'')} \cdot \tan \tfrac{1}{2}\,\phi_{\prime\prime}$$

We can also find the sides $IS_{\prime}$, $IS_{\prime\prime}$, of these spherical triangles; and then we have—

$$\triangle = \psi_{\prime\prime} - \psi_{\prime}$$

$$a_{\prime} = 90^\circ - IS_{\prime};\quad a_{\prime\prime} = IS_{\prime\prime} - 90^\circ.$$

And as a test of accuracy of the work we have $a_{\prime} + a_{\prime\prime} = \Sigma$.

EXAMPLE (Problem 1).

Let $l_{\prime} = 38^\circ$; $l_{\prime\prime} = 37^\circ$; $\omega = 1^\circ{}_{\prime\prime}\ 15'{}_{\prime\prime}\ 00''$; be the given latitudes and difference of longitude of the stations.

First then, to find the values of the normals $R_{\prime}$, $R_{\prime\prime}$, drawn at the stations S_o, S_{oo}, which terminate in the polar axis, we have the well known formula

$$R_{,} = \frac{a}{\sqrt{1 - e^2 \sin^2 l_{,}}}; \quad R_{,,} = \frac{a}{\sqrt{1 - e^2 \sin^2 l_{,,}}}$$

and we easily obtain

$$\log R_{,} = 7{\cdot}3212526296; \quad R_{,} = 20953309{\cdot}5777 \text{ feet};$$
$$\log R_{,,} = 7{\cdot}3212277292; \quad R_{,,} = 20952108{\cdot}2495 \text{ feet.}$$

We will now proceed to find the values of the small arcs $\delta_{,}$, $\delta_{,,}$, by means of formula 80. And as $R_{,} \cos l' - R_{,,} \cos l''$ enters in both numerators and denominators of the expressions, we shall first find its value. Thus :—

$\log R_{,} =$	$7{\cdot}3212526296$	$\log R_{,,} =$	$7{\cdot}321227292$
$\cos l' =$	$\bar{1}{\cdot}7893417987$	$\cos l'' =$	$\bar{1}{\cdot}7794630249$
	$7{\cdot}1105946083$		$7{\cdot}1006907541$

$$\text{antilogs} \begin{cases} 12900145{\cdot}48795 \\ 12609293{\cdot}51225 \end{cases}$$

$$\therefore R_{,} \cos l' - R_{,,} \cos l'' = 290851{\cdot}9757$$

and $\log (R_{,} \cos l' - R_{,,} \cos l'') = 5{\cdot}4636720181$

Now to find $\delta_{,}$ we have formula 80 or—

$$\tan \delta_{,} = \frac{e^2 (R_{,} \cos l' - R_{,,} \cos l'') \sin l'}{R_{,} - e^2 (R_{,} \cos l' - R_{,,} \cos l'') \cos l'}$$

$\log e^2 =$	$\bar{3}{\cdot}8315591974$	$\log e^2 =$	$\bar{3}{\cdot}8315591974$
	$5{\cdot}4636720182$		$5{\cdot}4636720182$
$\sin l' =$	$\bar{1}{\cdot}8965321441$	$\cos l' =$	$\bar{1}{\cdot}7893419787$
	$3{\cdot}1917633597$		$3{\cdot}0845731943$
		antilog $=$	$1214{\cdot}9913$
		but $R_{,} =$	$20953309{\cdot}5777$

$\therefore$ the value of the denominator $= 20952094{\cdot}5864$

and its log is $7{\cdot}3212274459$
$3{\cdot}1917633597$

$$\therefore \log \tan \delta_{,} = \bar{5}\ 8705359138$$

$$\therefore \delta_{,} = 0^{\circ}{}_{,,}\ 00'{}_{,,}\ 15''{\cdot}309501$$

To find $\delta_{,,}$ we have the formula 80 or—

$$\tan \delta_{,,} = \frac{e^2 (R_{,} \cos l' - R_{,,} \cos l'') \sin l''}{R_{,,} + e^2 (R_{,} \cos l' - R_{,,} \cos l'') \cos l''}$$

$$\begin{array}{rll} \log e^2 = & \bar{3}\cdot 8315591974 \\ & 5\cdot 4636720182 \\ \sin l'' = & \bar{1}\cdot 9023486165 \\ \hline & 3\cdot 1975798321 \end{array} \qquad \begin{array}{rll} \log e^2 = & \bar{3}\cdot 8315591974 \\ & 5\cdot 4636720182 \\ \cos l'' = & \bar{1}\cdot 7794630249 \\ \hline & 3\cdot 0746942397 \\ \text{antilog} = & 1187\cdot 6658 \\ = & 20952108\cdot 2495 \\ \hline \end{array}$$

$$\therefore \text{ value of denominator} = 20953295\cdot 9153$$

$$\begin{array}{rl} \text{its log} = & 7\cdot 3212523464 \\ & 3\cdot 1975798321 \\ \hline \therefore \log \tan \delta_{,,} = & \bar{5}\cdot 8763274857 \end{array}$$

$$\therefore \delta_{,,} = 0°_{,,}\ 00'_{,,}\ 15''\cdot 51503$$

To find the arcs L′ and L″, we have

$$L' = l' + \delta_{,} \qquad L'' = l'' - \delta_{,,}$$

$$\begin{array}{rl} l' = & 52° \\ \delta_{,} = & 0_{,,}\ 00'_{,,}\ 15''\cdot 30950 \\ \hline \therefore L' = & 52°_{,,} 00'_{,,}\ 15''\cdot 30950 \end{array} \qquad \begin{array}{rl} l'' = & 53° \\ \delta_{,,} = & 0_{,,}\ 00'_{,,} 15''\cdot 51503 \\ \hline \therefore L'' = & 52°_{,,} 59'_{,,} 44''\cdot 48497 \end{array}$$

These values are correct to the last or fifth decimals.

To find L′ we have also the formula 79 or—

$$\cot L' = (1 - e^2) \cot l' + e^2 \cdot \frac{R_{,,} \cos l''}{R_{,} \sin l'}$$

$$\begin{array}{rl} \log (1 - e^2) = & \bar{1}\cdot 9970432059 \\ \cot l' = & \bar{1}\cdot 8928098346 \\ \hline & \bar{1}\cdot 8898530405 \\ \text{antilog} = & 0\cdot 7759844892 \end{array} \qquad \begin{array}{rl} \log e^2 = & \bar{3}\cdot 8315591974 \\ \log R_{,,} = & 7\cdot 3212277292 \\ \cos l'' = & \bar{1}\cdot 7794630249 \\ \hline & 4\ 9322499515 \end{array}$$

$$\begin{array}{rl} \log R_{,} = & 7.3212526296 \\ \sin l' = & \bar{1}\cdot 8965321441 \\ \hline & 7\cdot 2177847737 \\ & 4\cdot 9322499515 \\ \hline & \bar{3}\cdot 7144651778 \end{array}$$

$$\begin{array}{rl} \text{antilog} = & 0\cdot 0051816154 \\ & 0\cdot 7759844892 \\ \hline \therefore \cot L' = & 0\cdot 7811661046 \end{array}$$

$$\therefore \log \cot L' = \bar{1}\cdot 8927433907$$

$$\therefore L' = 52°_{,,}\ 00'_{,,}\ 15''\cdot 3095$$

To find L″ we have formula 79 or—

$$\cot L'' = (1 - e^2) \cot l'' + e^2 \cdot \frac{R_{,} \cos l'}{R_{,,} \sin l''}$$

$$\begin{aligned} \log (1 - e^2) &= \bar{1}\cdot9970432059 \\ \cot l'' &= \bar{1}\cdot8771144084 \\ \hline & \bar{1}\cdot8741576143 \qquad \text{antilog} = 0\cdot7484410756 \end{aligned}$$

$$\begin{aligned} \log e^2 &= \bar{3}\cdot8315591974 \qquad & \log R_{,,} &= 7\cdot3212277292 \\ \log R_{,} &= 7\cdot3212526296 & \sin l'' &= \bar{1}\cdot9023486165 \\ \cos l' &= \bar{1}\cdot7893419787 & & 7\cdot2235763457 \\ & 4\cdot9421538057 \\ & 7\cdot2236012457 \\ & \bar{3}\cdot7185525600 & \text{antilog} &= 0\cdot0052309125'5 \\ & & & 0\cdot7484410756'5 \end{aligned}$$

$$\therefore \text{ nat} \cot L'' = 0\cdot7536719882$$

$$\therefore \log \cot L'' = \bar{1}\cdot8771823669, \text{ and } L'' = 52^{\circ}{}_{,,}\ 59'{}_{,,}\ 44''\cdot4867$$

the error of 0″·0018 being due to the insufficiency of the tables or to their inaccuracy in the 10th decimal places, &c.

Now, in each of the spherical triangles $S_{,}PD_{,,}$, $S_{,,}PD_{,,}$, $S_{,}PS_{,,,}$ we have the two sides and the included angle ω from which we can find the angles at their bases and also the bases.

To find the angles $A_{,,}$ $D_{,,,}$ and base $z_{,}$ of the triangle $S_{,}PD_{,,}$—

$$\begin{aligned} \cot \tfrac{1}{2}\omega &= 11\cdot9622253888 & \cot \tfrac{1}{2}\omega &= 11\cdot9622253888 \\ \cos \tfrac{1}{2}(L'' - l') &= 9\cdot9999836052 & \sin \tfrac{1}{2}(L'' - l') &= 7\cdot9389661700 \\ & 21\cdot9622089940 & & 19\cdot9011915588 \\ \cos \tfrac{1}{2}(L'' + l') &= 9\cdot7844684133 & \sin \tfrac{1}{2}(L'' + l') &= 9\cdot8994541209 \end{aligned}$$

$$\therefore \tan \tfrac{1}{2}(A_{,} + D_{,,}) = 12\cdot1777405807 \qquad \therefore \tan \tfrac{1}{2}(A_{,} - D_{,,}) = 10\cdot0017374379$$

$$\therefore \tfrac{1}{2}(A_{,} + D_{,,}) = 89^{\circ}{}_{,,}\ 37'{}_{,,}\ 10''\cdot133745$$

$$\therefore \tfrac{1}{2}(A_{,} - D_{,,}) = 45^{\circ}{}_{,,}\ 06'{}_{,,}\ 52''\cdot590185$$

$$\therefore A_{,} = 134^{\circ}{}_{,,}\ 44'{}_{,,}\ 02''\cdot72393$$

$$D_{,,} = 44^{\circ}{}_{,,}\ 30'{}_{,,}\ 17''\cdot54356$$

$$\sin l' = 9\cdot8965321441$$
$$\sin \omega = 8\cdot3387529285$$
$$18\cdot2352850726$$
$$\sin D_{\prime\prime} = 9\cdot8456993857$$
$$\therefore\quad \sin z_{\prime} = 8\cdot3895856869$$

$$\sin L'' = 9\cdot9023239980$$
$$\sin \omega = 8\cdot3387529285$$
$$18\cdot2410769265$$
$$\sin A_{\prime} = 9\cdot8514912397$$
$$\therefore\quad \sin z_{\prime} = 8\cdot3895856868$$

$$\therefore\quad z_{\prime} = 1^\circ{}_{\prime\prime}\ 24'{}_{\prime\prime}\ 18''\cdot8798$$

To find the angles $D_{\prime}$, $A_{\prime\prime}$, and base $z_{\prime\prime}$ of the triangle $A_{\prime\prime}PD_{\prime}$,—

$$\cot \tfrac{1}{2}\,\omega = 11\cdot9622253888$$
$$\cos \tfrac{1}{2}\,(l''-L') = 9\cdot9999836034$$
$$21\cdot9622089922$$
$$\cos \tfrac{1}{2}\,(l''+L') = 9\cdot7844261226$$
$$\therefore\quad \tan \tfrac{1}{2}\,(D_{\prime}+A_{\prime\prime}) = 12\cdot1777828696$$

$$\cot \tfrac{1}{2}\,\omega = 11\cdot9622253888$$
$$\sin \tfrac{1}{2}\,(l''-L') = 7\cdot9389910706$$
$$19\cdot9012164594$$
$$\sin \tfrac{1}{2}\,(l''+L') = 9\cdot8994790213$$
$$\therefore\quad \tan \tfrac{1}{2}\,(D_{\prime}-A_{\prime\prime}) = 10\cdot0017374381$$

$$\therefore\quad \tfrac{1}{2}\,(D_{\prime}+A_{\prime\prime}) = 89^\circ{}_{\prime\prime}\ 37'{}_{\prime\prime}\ 10''\cdot267152$$
$$\therefore\quad \tfrac{1}{2}\,(D_{\prime}-A_{\prime\prime}) = 45^\circ{}_{\prime\prime}\ 06'{}_{\prime\prime}\ 52''\cdot590233$$
$$\therefore\ D_{\prime} = 134^\circ{}_{\prime\prime}\ 44'{}_{\prime\prime}\ 02''\cdot857385$$
$$A_{\prime\prime} = 44^\circ{}_{\prime\prime}\ 30'{}_{\prime\prime}\ 17''\cdot676919$$

$$\sin l'' = 9\cdot9023486165$$
$$\sin \omega = 8\cdot3387529285$$
$$18\cdot2411015450$$
$$\sin D_{\prime} = 9\cdot8514909614$$
$$\because\ \sin z_{\prime\prime} = 8\cdot3896105836$$

$$\sin L' = 9\cdot8965573265$$
$$\sin \omega = 8\cdot3387529285$$
$$18\cdot2353102550$$
$$\sin A_{\prime\prime} = 9\cdot8456996715$$
$$\therefore\quad \sin z_{\prime\prime} = 8\cdot3896105835$$

$$\therefore\quad z_{\prime\prime} = 1^\circ{}_{\prime\prime}\ 24'{}_{\prime\prime}\ 19''\cdot169884$$

To find the angles $A_{\circ}$, $A_{\circ\circ}$, and base ν of the triangle $S_{\prime}PS_{\prime\prime}$,—

$$\cot \tfrac{1}{2}\,\omega = 11\cdot9622253888$$
$$\cos \tfrac{1}{2}\,(l''-l') = 9\cdot9999834631$$
$$21\cdot9622088519$$
$$\cos \tfrac{1}{2}\,(l''+l') = 9\cdot7844471278$$
$$\therefore\quad \tan\tfrac{1}{2}(A_{\circ}+A_{\circ\circ}) = 12\cdot1777617241$$

$$\cot \tfrac{1}{2}\,\omega = 11\cdot9622253888$$
$$\sin \tfrac{1}{2}\,(l''-l') = 7\cdot9408418596$$
$$19\cdot9030672484$$
$$\sin \tfrac{1}{2}\,(l''+l') = 9\cdot8994666546$$
$$\therefore\quad \tan\tfrac{1}{2}(A_{\circ}-A_{\circ\circ}) = 10\cdot0036005938$$

$$\therefore\quad \tfrac{1}{2}\,(A_{\circ}+A_{\circ\circ}) = 89^\circ{}_{\prime\prime}\ 37'{}_{\prime\prime}\ 10''\cdot20043$$
$$\therefore\quad \tfrac{1}{2}\,(A_{\circ}-A_{\circ\circ}) = 45^\circ{}_{\prime\prime}\ 14'{}_{\prime\prime}\ 15''\cdot02727$$
$$\therefore\quad A_{\circ} = 134^\circ{}_{\prime\prime}\ 51'{}_{\prime\prime}\ 25''\cdot22770$$
$$A_{\circ\circ} = 44^\circ{}_{\prime\prime}\ 22'{}_{\prime\prime}\ 55''\cdot17316$$

$$\begin{array}{rl} \sin l' = & 9{\cdot}8965321441 \\ \sin \omega = & 8{\cdot}3387529285 \\ \hline & 18{\cdot}2352850726 \\ \sin A_{\circ\circ} = & 9{\cdot}8447496921 \\ \hline \therefore \sin \nu = & 8{\cdot}3905353805 \end{array} \qquad \begin{array}{rl} \sin l'' = & 9{\cdot}9023486165 \\ \sin \omega = & 8{\cdot}3387529285 \\ \hline & 18{\cdot}2411015450 \\ \sin A_{\circ} = & 9{\cdot}8505661645 \\ \hline \therefore \sin \nu = & 8{\cdot}3905353805 \end{array}$$

$$\therefore \nu = 1^\circ{}_{\prime\prime}\, 24'{}_{\prime\prime}\, 29''\cdot 956648$$

To find the portions $\nu_{\prime\prime}$, $\nu_{\prime}$, into which ν is divided by the point O.

From the spherical triangles $S_{\prime\prime}OE_{\prime\prime}$, $S_{\prime}OE_{\prime}$, we have—

$$\sin \nu_{\prime\prime} \cdot \sin O = \sin \alpha_{\prime\prime}; \quad \sin \nu_{\prime} \cdot \sin O = \sin \alpha_{\prime};$$

and from these—

$$\frac{\sin \nu_{\prime\prime}}{\sin \nu_{\prime}} = \frac{\sin \alpha_{\prime\prime}}{\sin \alpha_{\prime}} = \frac{R_{\prime}}{R_{\prime\prime}};$$

and ∴ (see formulæ 27, 33, 34)—

$$\begin{array}{rl} \log R_{\prime} = & 7{\cdot}3212526296 \\ \log R_{\prime\prime} = & 7{\cdot}3212277292 \\ \hline \therefore \tan x = & 10{\cdot}0000249004 \end{array} \qquad \begin{array}{rl} \tan \tfrac{1}{2}\nu = & \bar{2}{\cdot}0895709833 \\ \tan (x-45^\circ) = & \bar{5}{\cdot}4573930282 \\ \hline \therefore \tan \tfrac{1}{2}(\nu_{\prime\prime}-\nu_{\prime}) = & \bar{7}{\cdot}5469640115 \end{array}$$

$$\therefore x = 45^\circ{}_{\prime\prime}\, 00'{}_{\prime\prime}\, 05''{\cdot}91314 \qquad \therefore \tfrac{1}{2}(\nu_{\prime\prime}-\nu_{\prime}) = 0^\circ{}_{\prime\prime}\, 00'{}_{\prime\prime}\, 00''{\cdot}072776$$

$$\text{But } \tfrac{1}{2}(\nu_{\prime\prime}+\nu_{\prime}) = 0^\circ{}_{\prime\prime}\, 42'{}_{\prime\prime}\, 14''{\cdot}978324$$

$$\therefore \nu_{\prime\prime} = 0^\circ{}_{\prime\prime}\, 42'{}_{\prime\prime}\, 15''{\cdot}051100$$

$$\nu_{\prime} = 0^\circ{}_{\prime\prime}\, 42'{}_{\prime\prime}\, 14''{\cdot}905548$$

To find the angles $\Omega_{\prime}$, $\Omega_{\prime\prime}$, which a plane parallel to the two normals makes with the normal chordal planes—

$$\Omega_{\prime} = A_{\circ} - A_{\prime} = 0^\circ{}_{\prime\prime}\, 07'{}_{\prime\prime}\, 22''{\cdot}50377$$

$$\Omega_{\prime\prime} = A_{\prime\prime} - A_{\circ\circ} = 0^\circ{}_{\prime\prime}\, 07'{}_{\prime\prime}\, 22''{\cdot}50377$$

∴ we have in actual practice (as has been already demonstrated) $\Omega_{\prime} = \Omega_{\prime\prime}$; and we may write Ω to represent their common value.

To find the angles $\alpha_{\prime}$, $\alpha_{\prime\prime}$, of depression of the chord below the tangent planes at the stations $S_{\circ}$, $S_{\circ\circ}$, we have—

$$\begin{array}{rl} \tan \alpha_{\prime} = & \tan \nu_{\prime} \cdot \cos \Omega \\ \tan \nu_{\prime} = & 8{\cdot}0895585138 \\ \cos \Omega = & 9{\cdot}9999990005 \\ \hline \therefore \tan \alpha_{\prime} = & 8{\cdot}0895575143 \end{array} \qquad \begin{array}{rl} \tan \alpha_{\prime\prime} = & \tan \nu_{\prime\prime} \cdot \cos \Omega \\ \tan \nu_{\prime\prime} = & 8{\cdot}0895834524 \\ \cos \Omega = & 9{\cdot}9999990005 \\ \hline \therefore \tan \alpha_{\prime\prime} = & 8{\cdot}0895824529 \end{array}$$

$$\therefore \alpha_{\prime} = 0^\circ{}_{\prime\prime}\, 42'{}_{\prime\prime}\, 14''{\cdot}899714 \qquad \therefore \alpha_{\prime\prime} = 0^\circ{}_{\prime\prime}\, 42'{}_{\prime\prime}\, 15''{\cdot}045266$$

$$\therefore \Sigma = \alpha_{\prime} + \alpha_{\prime\prime} = 1^\circ{}_{\prime\prime}\, 24'{}_{\prime\prime}\, 29''{\cdot}94498$$

To find the length of k the chord connecting the stations. We have—

$$k = \frac{R_{,,} \cos l_{,,} \sin \omega}{\sin A_{,} \cos \alpha_{,}} \qquad k = \frac{R_{,} \cos l_{,} \sin \omega}{\sin A_{,,} \cos \alpha_{,,}}$$

$$\begin{array}{lll} \log R_{,,} = 7{\cdot}3212277292 & & \log R_{,} = 7{\cdot}3212526296 \\ \cos l_{,,} = \bar{1}{\cdot}9023486165 & & \cos l_{,} = \bar{1}{\cdot}8965321441 \\ \sin \omega = \bar{2}{\cdot}3387529285 & & \sin \omega = \bar{2}{\cdot}3387529285 \\ \quad 5{\cdot}5623292745 & & \quad 5{\cdot}5565377022 \\ \sin A_{,} = \bar{1}{\cdot}8514912398 & & \sin A_{,,} = \bar{1}{\cdot}8456996715 \\ \cos \alpha_{,} = \bar{1}{\cdot}9999672028 & & \cos \alpha_{,,} = \bar{1}{\cdot}9999671990 \\ \quad \bar{1}{\cdot}8514584426 & & \quad \bar{1}{\cdot}8456668705 \\ \therefore \log k = 5{\cdot}7108708319 & & \therefore \log k = 5{\cdot}7108708317 \end{array}$$

$$\log k = 5{\cdot}7108708318$$

$$\therefore k = 513890{\cdot}787$$

To find the length of the geodesic arc s connecting the stations—

$$s = \frac{k \cdot \Sigma \cdot \sin 1''}{2 \cdot \sin \frac{1}{2} \Sigma}$$

$$\begin{array}{lll} \log k = 5{\cdot}7108708318 & & \log 2 = 0{\cdot}3010299957 \\ \log \Sigma = 3{\cdot}7050032463 & & \sin \frac{1}{2} \Sigma = \bar{2}{\cdot}0895371846 \\ \sin 1'' = \bar{6}{\cdot}6855748668 & & \quad \bar{2}{\cdot}3095671803 \\ \quad 4{\cdot}1014489449 & & \\ \quad \bar{2}{\cdot}3905671803 & & \\ \therefore \log s = 5{\cdot}7108817646 & & \therefore s = 513903{\cdot}723718 \text{ feet.} \end{array}$$

To find the arcs $OE_{,}$, $OE_{,,}$, or $\gamma_{,}$ $\gamma_{,,}$, whose sum $E_{,}E_{,,}$ is the measure of the angle $\triangle$. We have—

$$\begin{array}{lll} \sin \gamma_{,} = \sin \nu_{,} \sin \Omega & & \sin \gamma_{,,} = \sin \nu_{,,} \sin \Omega \\ \sin \nu_{,} = 8{\cdot}0895257164 & & \sin \nu_{,,} = 8{\cdot}0895506513 \\ \sin \Omega = 7{\cdot}3314915049 & & \sin \Omega = 7{\cdot}3314915049 \\ \therefore \sin \gamma_{,} = 5{\cdot}4210172213 & & \sin \gamma_{,,} = 5{\cdot}4210421562 \\ \therefore \gamma_{,} = 0^{\circ}{}_{,,}\, 00'{}_{,,}\, 05'' \cdot 438039 & & \therefore \gamma_{,} = 0^{\circ}{}_{,,}\, 00'{}_{,,}\, 05'' \cdot 438352 \end{array}$$

$$\therefore \triangle = 0^{\circ}{}_{,,}\, 00'{}_{,,}\, 10'' \cdot 876391$$

To find the arcs $e_{,}$, $f_{,}$, whose sum $= \delta_{,}$. Since the pencil I ($S_{,}S_{,,}OP$) is harmonic, we have—

$$\tan \tfrac{1}{2} (f_{,} - e_{,}) = \frac{\tan^2 \frac{1}{2} \delta_{,}}{\tan \frac{1}{2} (L' + l')}; \quad \tfrac{1}{2} (f_{,} + e_{,}) = \tfrac{1}{2} \delta_{,}$$

And to find the arcs $e_{,,}$, $f_{,,}$, whose sum $= \delta_{,,}$; we have—

$$\tan \tfrac{1}{2} (e_{,,} - f_{,,}) = \frac{\tan^2 \frac{1}{2} \delta_{,,}}{\tan \frac{1}{2} (L'' + l')} ; \quad \tfrac{1}{2} (e_{,,} + f_{,,}) = \tfrac{1}{2} \delta_{,,}$$

From these we easily obtain the values—

$$e_{,,} = 7{\cdot}75773 \qquad f_{,,} = 7{\cdot}75729$$
$$e_{,} = 7{\cdot}65453 \qquad f_{,} = 7{\cdot}65497$$

In the spherical triangle $F_{,}PF_{,,}$, we know the values of the sides and included angle ω; and applying the usual formulæ we find—

$$\text{angle } F_{,} = 134^\circ{}_{,,}\ 44'{}_{,,}\ 02'' \cdot 79079$$
$$\text{angle } F_{,,} = 44^\circ{}_{,,}\ 30'{}_{,,}\ 17'' \cdot 61004$$
$$\text{arc } F_{,}F_{,,} = 1^\circ{}_{,,}\ 24'{}_{,,}\ 19'' \cdot 02484 = \tfrac{1}{2} (z_{,} + z_{,,})$$
$$\therefore\ F_{,} = \tfrac{1}{2} (A_{,} + D_{,}) \text{ to within } 0''{\cdot}0001$$
$$\therefore\ F_{,,} = \tfrac{1}{2} (A_{,,} + D_{,,}) \text{ to within } 0''{\cdot}0002$$

We may also observe that—

$$D_{,} - A_{,} = 0''{\cdot}13345 ; \quad A_{,,} - D_{,,} = 0''{\cdot}13336$$
$$\therefore\ D_{,} - A_{,} = A_{,,} - D_{,,} \text{ to within } 0''{\cdot}0001$$

☞ In the "Account of the Principal Triangulation of Great Britain and Ireland," the following formulæ are given—

$$D_{,} - A_{,} = \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \cos^2 l_{,,} \sin 2 A_{,,} \cdot z_{,,}^2 \cdot \sin 1''$$

$$D_{,,} - A_{,,} = \tfrac{1}{4} \cdot \frac{e^2}{1 - e^2} \cdot \cos^2 l_{,} \sin 2 A_{,} \cdot z_{,}^2 \cdot \sin 1''$$

In working out these expressions with respect to the present examples we have—

$\log \frac{1}{4}$	$= \bar{1}{\cdot}3979400087$	$\log \frac{1}{4}$	$= \bar{1}{\cdot}3979400087$
$\log \frac{e^2}{1 - e^2}$	$= \bar{3}{\cdot}8345159915$	$\log \frac{e^2}{1 - e^2}$	$= \bar{3}{\cdot}8345159915$
$\cos^2 l_{,,}$	$= \bar{1}{\cdot}8046972330$	$\cos^2 l_{,}$	$= \bar{1}{\cdot}7930642882$
$\sin 2 A_{,,}$	$= \bar{1}{\cdot}9999812911$	$\sin 2 A_{,}$	$= \bar{1}{\cdot}9997379520$
$\log z_{,,}^2$	$= 7{\cdot}4081585260$	$\log z_{,}^2$	$= 7{\cdot}4081087226$
$\sin 1''$	$= \bar{6}{\cdot}6855748668$	$\sin 1''$	$= \bar{6}{\cdot}6855748668$
$\therefore \log (D_{,} - A_{,})$	$= \bar{1}{\cdot}1308679171$	$\therefore \log (A_{,,} - D_{,,})$	$= \bar{1}{\cdot}1189418298$

$$\therefore\ D_{,} - A_{,} = 0''{\cdot}1352 \text{ which is too great by } 0''{\cdot}002$$
$$A_{,,} - D_{,,} = 0''{\cdot}1315 \text{ which is too small by } 0''{\cdot}002$$

We may also observe that in all cases in which the greater azimuth $A_{,}$ is less than 90°, the second of the above formulæ would intimate that $D_{\prime\prime}$ is greater than $A_{\prime\prime\prime}$, which we know to be erroneous. And when $A_{\prime} = 90°$ it intimates that $D_{\prime\prime} = A_{\prime\prime\prime}$, which is also erroneous.

☞ In order to shew the extent to which a change in the assumed values of the earth's polar and equatorial radii can effect the results of geodetic computations, I give the following columns of results, worked out with 7 place logs.—

FOR THE LATEST CONSTANTS.	FOR CONSTANTS FORMERLY USED.
$\left\{ \begin{array}{l} a = 20926348 \\ b = 20855233 \end{array} \right\}$	$\left\{ \begin{array}{l} a = 20923713 \\ b = 20853810 \end{array} \right\}$
$A_{\circ} = 134°_{\prime\prime}\ 51'_{\prime\prime}\ 25''{\cdot}225$	$A_{\circ}$ = same as before
$A_{\circ\circ} = 44_{\prime\prime}\ 22_{\prime\prime}\ 55 \cdot 177$	$A_{\circ\circ}$ = „ „ „
$A_{\prime} = 134°_{\prime\prime}\ 44_{\prime\prime}\ 03 \cdot 683$	$A_{\prime} = 134°_{\prime\prime}\ 44'_{\prime\prime}\ 10''{\cdot}647$
$A_{\prime\prime} = 44_{\prime\prime}\ 30_{\prime\prime}\ 16 \cdot 718$	$A_{\prime\prime} = 44°_{\prime\prime}\ 30_{\prime\prime}\ 09 \cdot 754$
$\Omega = 0_{\prime\prime}\ 07_{\prime\prime}\ 21 \cdot 541$	$\Omega = 0_{\prime\prime}\ 07_{\prime\prime}\ 14 \cdot 577$
$\nu = 1_{\prime\prime}\ 24_{\prime\prime}\ 29 \cdot 956$	ν = same as before
$\Sigma = 1_{\prime\prime}\ 24_{\prime\prime}\ 29 \cdot 945$	Σ = „ „ „
$\alpha_{\prime} = 0_{\prime\prime}\ 42_{\prime\prime}\ 14 \cdot 900$	$\alpha_{\prime} = 0°_{\prime\prime}\ 42'_{\prime\prime}\ 14 \cdot 901$
$\alpha_{\prime\prime} = 0_{\prime\prime}\ 42_{\prime\prime}\ 15 \cdot 045$	$\alpha_{\prime\prime} = 0_{\prime\prime}\ 42'_{\prime\prime}\ 15 \cdot 045$
$\Delta = 0_{\prime\prime}\ 00_{\prime\prime}\ 10 \cdot 852$	$\Delta = 0_{\prime\prime}\ 00_{\prime\prime}\ 10 \cdot 681$
$s = 513905{\cdot}8$ feet	$s = 513847{\cdot}7$ feet

The increase in $A_{\prime}$ is equal to the decrease in $A_{\prime\prime}$, and the whole amount $6''{\cdot}9$ of such increase or decrease is owing to the change in the ratio of a to b, and not to their absolute magnitudes. This shews that if the assumed value $\frac{a}{b}$ be not suitable to the locality of the survey, there must of necessity be discrepancies between the azimuths as found by direct observation and computations, in closing work carried on by means of two series of stations. We see also that the values of s differ by about 58 feet in an arc of 97 miles, owing to the change in the values of a and b.

Example (Problem 2).

Case 1.

Given the latitude $l_{\prime} = 38°$; the azimuth $A_{\prime} = 134°_{\prime\prime}\ 44'_{\prime\prime}\ 02''{\cdot}72393$; and the length of the geodesic arc $s = 513903{\cdot}7237$ feet; to find the difference of longitude ω, the latitude $l_{\prime\prime}$ the azimuth $A_{\prime\prime}$, &c.

To find $z_{,}$ we have (from the "Account of the Principal Triangulation of Great Britain and Ireland") the formula—

$$\log z_{,} = \log\left(\frac{s}{R_{,} \sin 1''}\right) + 0{\cdot}0004862 \times \sin^2(\triangle l') \cdot \sin^2 l'$$

in which $(\triangle l')$ represents any close approximate to the difference of the given and unknown latitudes, so as to have the first three or four decimal places in the expression log $(\sin^2 \triangle l')$ correct.

In the present example we know that $\triangle l' = 1°$ nearly, and ∴ to find $z_{,}$—

$$\log(0{\cdot}0004862) = \bar{4}{\cdot}6868 \qquad \log R_{,} = 7{\cdot}3212526296$$
$$\sin^2(\triangle l') = 6{\cdot}4837 \qquad \sin 1'' = \bar{6}{\cdot}6855748668$$
$$\sin^2 l' = \bar{1}{\cdot}7931 \qquad 2{\cdot}0068274964$$
$$\log s = 5{\cdot}7108817646$$
$$\text{antilog} = 919{\cdot}6 \qquad 3{\cdot}7040542682$$
$$919{\cdot}6$$
$$\therefore \log z_{,} = 3{\cdot}7040543601$$
$$\therefore z_{,} = 1°_{,,}\ 24'_{,,}\ 18''{\cdot}8798$$

Were we to use the more simple formulæ—

$$z_{,} = \frac{s}{R_{,} \cdot \sin 1''}$$

we evidently have—

$$\log z_{,} = 3{\cdot}7040542682$$

$$\therefore \quad z_{,} = 5058''{\cdot}8785 \doteq 1°_{,,}\ 24'_{,,}\ 18''{\cdot}8785,$$

which is too small by about $0''{\cdot}001$ only. And since the $0''{\cdot}001$ part of one second represents not more than an error of $\frac{1}{10}$ of a foot in the whole length of the arc $s = 97$ miles; ∴ it is evident that in all cases we can safely find $z_{,}$ by means of this formula.

Now knowing $A_{,,}$ l', $z_{,}$, in the spherical triangle $S_{,}PD_{,,,}$ we can find the angles ω, $D_{,,,}$ and the side L'' by the usual forms—

$$\tan \tfrac{1}{2}(D_{,,} + \omega) = \frac{\cos \frac{1}{2}(l' - z_{,})}{\cos \frac{1}{2}(l' + z_{,})} \cot \tfrac{1}{2} A_{,}$$

$$\tan \tfrac{1}{2}(D_{,,} - \omega) = \frac{\sin \frac{1}{2}(l' - z_{,})}{\sin \frac{1}{2}(l' + z_{,})} \cot \tfrac{1}{2} A_{,}$$

$$\cot \tfrac{1}{2} A_, = 9{\cdot}6200681684 \qquad \cot \tfrac{1}{2} A_, = 9{\cdot}6200681684$$
$$\cos \tfrac{1}{2} (l'-z_,) = 9{\cdot}9562174764 \qquad \sin \tfrac{1}{2} (l'-z_,) = 9{\cdot}6307496490$$
$$19{\cdot}5762856448 \qquad 19{\cdot}2508178174$$
$$\cos \tfrac{1}{2} (l'+z_,) = 9{\cdot}9510220423 \qquad \sin \tfrac{1}{2} (l'+z_,) = 9{\cdot}6525942988$$
$$\therefore \tan \tfrac{1}{2} (D_{,,}+\omega) = 9{\cdot}6252636025 \quad \therefore \tan \tfrac{1}{2} (D_{,,}-\omega) = 9{\cdot}5982235186$$

$$\tfrac{1}{2} (D_{,,} + \omega) = 22^\circ{}_{,,}\ 52'{}_{,,}\ 38''{\cdot}7711$$
$$\therefore \tfrac{1}{2} (D_{,,} - \omega) = 21^\circ{}_{,,}\ 37'{}_{,,}\ 38''{\cdot}7719$$
$$\therefore D_{,,} = 44^\circ{}_{,,}\ 30'{}_{,,}\ 17''{\cdot}5430$$
$$\omega = 1^\circ{}_{,,}\ 14'{}_{,,}\ 59''{\cdot}9992$$

☞ This case, in which the given latitude is *greater* than the sought latitude, is made known to us by $A_,$ being greater than the angle $D_{,,}$.

To find L''—

$$\sin z_, = 8{\cdot}3895856868 \qquad \sin l' = 9{\cdot}8965321441$$
$$\sin A_, = 9{\cdot}8514912398 \qquad \sin A_, = 9{\cdot}8514912398$$
$$18{\cdot}2410769266 \qquad 19{\cdot}7480233839$$
$$\sin \omega = 8{\cdot}3387529285 \qquad \sin D_{,,} = 9{\cdot}8456993857$$
$$\therefore \sin L'' = 9{\cdot}9023239981 \qquad \therefore \sin L'' = 9{\cdot}9023239982$$
$$\because L'' = 52^\circ{}_{,,}\ 59'{}_{,,}\ 44''{\cdot}4850$$

or to find L'' we may use the formula—

$$\tan \tfrac{1}{2} (L'' - l') = \frac{\sin \tfrac{1}{2} (A_, - D_{,,})}{\sin \tfrac{1}{2} (A_, + D_{,,})} \cdot \tan \tfrac{1}{2} z_,$$

To find $\delta_{,,}$ we have the approximate formula 84—

$$\delta_{,,} = \frac{e^2}{1 - e^2} \cdot \sin L'' \sin \tfrac{1}{2} (L'' + l') \cdot (L'' - l')$$

or the more closely approximate formula 83—

$$\sin \delta_{,,} = \frac{2 \cdot e^2 \cdot \sin \tfrac{1}{2} (L''+l') \sin \tfrac{1}{2} (L'' - l') \sin L''}{(1 - e^2) - 2 \cdot e^2 \cdot \sin \tfrac{1}{2} (L''+l') \sin \tfrac{1}{2} (L'' - l') \cos L''}$$

$$\log \frac{e^2}{1 - e^2} = \bar{3}{\cdot}8345160$$
$$\sin L'' = \bar{1}{\cdot}9023240$$
$$\sin \tfrac{1}{2} (L'' + l') = \bar{1}{\cdot}8994540$$
$$\log (L'' - l') = 3{\cdot}5544268$$
$$\therefore \log \delta_{,,} = 1{\cdot}1907208$$
$$\delta_{,,} = 0^\circ{}_{,,}\ 00'{}_{,,}\ 15''{\cdot}5139$$

$$\log 2 = 0{\cdot}3010300$$
$$\log e^2 = \bar{3}{\cdot}8315592$$
$$\sin \tfrac{1}{2}(L'' + l') = \bar{1}{\cdot}8994540$$
$$\sin \tfrac{1}{2}(L'' - l') = \bar{3}{\cdot}9389661$$

$$\bar{5}{\cdot}9710093 \quad . \quad . \quad . \quad . \quad . \quad \bar{5}{\cdot}9710093$$
$$\cos L'' = \bar{1}{\cdot}7795064 \qquad \sin L'' = \bar{1}{\cdot}9023240$$
$$\bar{5}{\cdot}7505157 \qquad \bar{5}{\cdot}8733333$$
$$\text{antilog} = 0{\cdot}000056300 \qquad \bar{1}{\cdot}9970186$$
$$1 - e^2 = 0{\cdot}993214854 \qquad \therefore \sin \delta_{,,} = \bar{5}{\cdot}8763147$$
$$0{\cdot}993158554 \qquad \therefore \delta_{,,} = 0°_{,,}\ 00'_{,,}\ 15''{\cdot}5146$$
$$\text{its log} = \bar{1}{\cdot}9970186$$

Then to find l'', and $l_{,,}$, we have—

$$l'' = L'' + \delta_{,,}; \quad l_{,,} = 90° - l''$$

$\therefore$ By first value of $\delta_{,,}$ we find $l_{,,} = 37°_{,,}\ 00'_{,,}\ 00''{\cdot}0019$
„ second „ „ $l_{,,} = 37°_{,,}\ 00'_{,,}\ 00''{\cdot}0004$

To find $A_{,,}$, we have—

$$A_{,,} - D_{,,} = \sin D_{,,} \tan \tfrac{1}{2} z_{,} \cdot \delta_{,,}$$

$$\sin D_{,,} = \bar{1}{\cdot}8456994 \qquad \therefore A_{,,} - D_{,,} = 0°_{,,}\ 00'_{,,}\ 00''{\cdot}13336$$
$$\tan \tfrac{1}{2} z_{,} = \bar{2}{\cdot}0886210 \qquad \text{but } D_{,,} = 44°_{,,}\ 30'_{,,}\ 17''{\cdot}5430$$
$$\log \delta_{,,} = 1{\cdot}1907207 \qquad \therefore A_{,,} = 44°_{,,}\ 30'_{,,}\ 17''{\cdot}6764$$
$$\therefore \log (A_{,,} - D_{,,}) = \bar{1}{\cdot}1250411$$

☞ In the "Account of the Principal Triangulation of Great Britain and Ireland" (see pages 247, 249, 676, of that work) there is given what is considered the most accurate method of solving this problem. The values of $z_{,}$, ω, $D_{,,}$, are there found as in the present paper, but the azimuth $A_{,,}$ and latitude $l_{,,}$ are determined otherwise: thus—

To find $A_{,,}$ the erroneous formula 96 is used, which gives $\zeta = A_{,,} - D_{,,} = 0''{\cdot}1315$ instead of $0''{\cdot}1334$.

Then to find $l_{,,}$ the following formula is given—

$$l_{,} - l_{,,} = \frac{s}{\rho \cdot \sin 1''} \cdot \frac{\sin \frac{1}{2}(A_{,} - A_{,,} + \zeta)}{\sin \frac{1}{2}(A_{,} + A_{,,} + \zeta)} \cdot \left\{ 1 + \frac{z_{,}^2}{12} \cdot \cos^2 \tfrac{1}{2}(A_{,} - A_{,,}) \sin^2 1 \right\}$$

in which ρ is the radius of curvature for the meridian for the mean between the known and unknown latitudes, and in which—

$$\tfrac{1}{2}(A_{,} - A_{,,} + \zeta) = \tfrac{1}{2}(A_{,} - D_{,,})$$
$$\tfrac{1}{2}(A_{,} + A_{,,} + \zeta) = \tfrac{1}{2}(A_{,} + D_{,,}).$$

The value of $l_{,} - l_{,,}$ as computed from the above is—

$$l_{,} - l_{,,} = 3600''\cdot0057 = 1^{\circ}{}_{,,}\ 00'{}_{,,}\ 00''\cdot0057$$
$$\therefore\ l_{,,} = 36^{\circ}{}_{,,}\ 59'{}_{,,}\ 59''\cdot9943,$$

which is nearly $0''\cdot006$ in error, when by the method followed in this paper the error amounts only to about $0''\cdot0004$.

It may perhaps be proper to observe that in the example under consideration we have in reality—

$$\tfrac{1}{2}(A_{,} + A_{,,} - \zeta) = \tfrac{1}{2}(A_{,} + D_{,,})$$

so that the fact of the expression for $l_{,} - l_{,,}$, being written as above shews that its author considered $A_{,,}$ to be less than $D_{,,}$: however, we know that $A_{,,}$ must be greater than $D_{,,}$.

Example (Problem 2).

Case 2.

Given the latitude $l_{,,} = 37^{\circ}$; the azimuth $A_{,,} = 44^{\circ}{}_{,,}\ 30'{}_{,,}\ 17''\cdot67692$; and the length of the geodesic arc $s = 513903\cdot7237$ feet: to find ω, $l_{,}$, and $A_{,}$, &c.

To find the arc $z_{,,}$, we have—

$$\log z_{,,} = \log \frac{s}{R_{,,} \sin 1} + 0\cdot0004862 \times \sin^2(\Delta l'') \sin^2 l''$$

in which $\Delta l''$ is the nearest approximate which we can easily find to the difference of the known and unknown latitudes. In the present case we know that $\Delta l''$ is nearly 1°.

$\log(0\cdot0004862)$ =	$\bar{4}\cdot6868$	$\log R_{,,}$ =	$7\cdot3212277292$
$\log \sin^2(\Delta l'')$ =	$6\cdot4837$	$\sin 1''$ =	$\bar{6}\cdot6855748668$
$\sin^2 l''$ =	$\bar{1}\cdot8047$		$2\cdot0068025960$
	$2\cdot9752$	$\log s$ =	$5\cdot7108817646$
antilog = $944\cdot5$			$3\cdot7040791686$
			944
		$\therefore \log z_{,,}$ =	$3\cdot7040792630$

$$\therefore\ z_{,,} = 5059''\cdot16988 = 1^{\circ}{}_{,,}\ 24'{}_{,,}\ 19''\cdot16988$$

Were we to use the simpler formula—

$$\log z_{\prime\prime} = \log \frac{s}{R_{\prime\prime} \sin 1''};$$

then, obviously, we have—

$$\log z_{\prime\prime} = 3\cdot7040792, \quad \text{and} \therefore \quad z_{\prime\prime} = 1^{\circ}{}_{\prime\prime}\ 24'{}_{\prime\prime}\ 19''\cdot1687$$

which is 0″·0011 too small.

To find $D_{\prime}$ and ω, we have—

$$\tan \tfrac{1}{2}(D_{\prime} + \omega) = \frac{\cos \frac{1}{2}(l'' - z_{\prime\prime})}{\cos \frac{1}{2}(l'' + z_{\prime\prime})} \cdot \cot \tfrac{1}{2} A_{\prime\prime}$$

$$\tan \tfrac{1}{2}(D_{\prime} - \omega) = \frac{\sin \frac{1}{2}(l'' - z_{\prime\prime})}{\sin \frac{1}{2}(l'' + z_{\prime\prime})} \cdot \cot \tfrac{1}{2} A_{\prime\prime}$$

$$\begin{aligned}
\cot \tfrac{1}{2} A_{\prime\prime} &= 10\cdot3881059553 \\
\cos \tfrac{1}{2}(l'' - z_{\prime\prime}) &= 9\cdot9544060605 \\
&\ 20\cdot3425120158 \\
\cos \tfrac{1}{2}(l'' + z_{\prime\prime}) &\ 9\cdot9490947477 \\
\therefore \tan \tfrac{1}{2}(D_{\prime} + \omega) &= 10\cdot3934172681
\end{aligned}$$

$$\begin{aligned}
\cot \tfrac{1}{2} A_{\prime\prime} &= 10\cdot3881059553 \\
\sin \tfrac{1}{2}(l'' - z_{\prime\prime}) &= 9\cdot6386781718 \\
&\ 20\cdot0267841271 \\
\sin \tfrac{1}{2}(l'' + z_{\prime\prime}) &= 9\cdot6600485181 \\
\therefore \tan \tfrac{1}{2}(D_{\prime} - \omega) &= 10\cdot3667356090
\end{aligned}$$

$$\therefore \tfrac{1}{2}(D_{\prime} + \omega) = 67^{\circ}{}_{\prime\prime}\ 59'{}_{\prime\prime}\ 31''\cdot4286$$

$$\therefore \tfrac{1}{2}(D_{\prime} - \omega) = 66^{\circ}{}_{\prime\prime}\ 44'{}_{\prime\prime}\ 31''\cdot4287$$

$$\therefore D_{\prime} = 134^{\circ}{}_{\prime\prime}\ 44'{}_{\prime\prime}\ 02'\cdot8573$$

$$\omega = 1^{\circ}{}_{\prime\prime}\ 15'{}_{\prime\prime}\ 00''\cdot0001$$

☞ This case in which the given latitude is *less* than the sought latitude, is made known to us by the given azimuth $A_{\prime\prime}$ being less than the computed angle $D_{\prime}$.

To find L′,—

$$\begin{aligned}
\sin z_{\prime\prime} &= 8\cdot3896105836 & \sin l'' &= 9\cdot9023486165 \\
\sin A_{\prime\prime} &= 9\cdot8456996715 & \sin A_{\prime\prime} &= 9\cdot8456996715 \\
&\ 18\cdot2353102551 & &\ 19\cdot7480482880 \\
\sin \omega &= 8\cdot3387529285 & \sin D_{\prime} &= 9\cdot8514909614 \\
\therefore \sin L' &= 9\cdot8965573266 & \therefore \sin L' &= 9\cdot8965573266
\end{aligned}$$

$$\therefore L' = 52^{\circ}{}_{\prime\prime}\ 00'{}_{\prime\prime}\ 15''\cdot3097$$

To find L′ we can also use the formula—

$$\tan \tfrac{1}{2} (l'' - L') = \frac{\sin \frac{1}{2} (D_, - A_{,,})}{\sin \frac{1}{2} (D_, + A_{,,})} \cdot \tan \tfrac{1}{2} z_{,,}$$

To find $\delta_{,}$, we have—

$$\log \frac{e^2}{1 - e^2} = \bar{3}\cdot 83451$$

$$\sin L' = \bar{1}\cdot 89655 \qquad \therefore \delta_, = 0^\circ_{,,}\, 00'_{,,}\, 15''\cdot 3098$$

$$\sin \tfrac{1}{2} (l'' + L') = \bar{1}\cdot 89946 \quad \therefore l' = L' - \delta_, = 51^\circ_{,,}\, 59'_{,,}\, 59''\cdot 9999$$

$$\log (l'' - L') = \underline{3\cdot 55445}$$

$$\therefore \log \delta_, = 1\cdot 18497 \qquad \therefore l_, = 38^\circ_{,,}\, 00'_{,,}\, 00''\ 0001$$

To find $A_{,}$, we have—

$$D_, - A_, = \sin D_, \tan \tfrac{1}{2} z_{,,} \cdot \delta_{,,}$$

$$\sin D_, = \bar{1}\cdot 85149$$

$$\tan \tfrac{1}{2} z_, = \bar{2}\cdot 08865 \qquad \therefore D_, - A_, = 0^\circ_{,,}\, 00'_{,,}\, 00''\cdot 1334$$

$$\log \delta_{,,} = \underline{1\cdot 18497} \qquad \text{But } D_, = \underline{134_{,,}\ 44_{,,}\ 02\ \cdot 8573}$$

$$\therefore \log (D_, - A_,) = \bar{1}\cdot 12511 \qquad \therefore A_, = 134^\circ_{,,}\, 44'_{,,}\, 02''\cdot 7239$$

☞ In the "Account of the Principal Triangulation of Great Britain and Ireland" the formula from which to find $l_,$ is—

$$l_, - l_{,,} = \frac{s}{\rho \cdot \sin 1''} \cdot \frac{\sin \frac{1}{2} (D_, - A_{,,})}{\sin \frac{1}{2} (D_, + A_{,,})} \cdot \left\{ 1 + \frac{z^2_{,,}}{12} \cdot \cos^2 \tfrac{1}{2} (A_, - A_{,,}) \sin^2 1'' \right\}$$

and the resulting value of $l_, - l_{,,} = 1^\circ_{,,}\, 00'_{,,}\, 00''\cdot 0059$
$\therefore l_, = 38^\circ_{,,}\, 00'_{,,}\, 00''\cdot 0059$ which is too great by $0''\cdot 006$, while by the method in this paper the error is only $0''\cdot 0001$.

In the treatise on "Geodesy" in Spon's Dictionary of Engineering, the unknown latitudes in the first and second cases of the problem are determined by means of the formulæ—

$$l_, - l_{,,} = \left\{ -\frac{s \cdot \cos A_,}{R_, \sin 1''} + \frac{s^2 \cdot \sin^2 A_, \tan l_,}{2 \cdot R^2_, \cdot \sin 1''} \right\} (1 + e^2 \cdot \cos^2 l_,)$$

$$l_, - l_{,,} = \left\{ +\frac{s \cdot \cos A_{,,}}{R_{,,} \cdot \sin 1''} - \frac{s^2 \cdot \sin^2 A_{,,} \tan l_{,,}}{2 \cdot R^2_{,,} \cdot \sin 1''} \right\} (1 + e^2 \cdot \cos^2 l_{,,})$$

from which we find $l_{,} - l_{,,} = 3600\,091$
and $l_{,} - l_{,,} = 3600{\cdot}632$; giving an error of $0''{\cdot}1$ in the first case, and an error of $0''{\cdot}6$ in the second case.

In Chambers' "Practical Mathematics" the formulæ differ from the above in having the factors $(1 + e^2 \cdot \cos l_{,})$, $(1 + e^2 \cdot \cos^2 l_{,,})$, replaced by $(1 + 2\,\epsilon \cdot \cos^2 l_{,,})$ and $(1 + 2\,\epsilon \cdot \cos^2 l_{,})$ which are greater; and ∴ obviously the results must be the more erroneous.

Their method of finding the difference of longitude is by means of the formula

$$\omega = \frac{s \cdot \sin A_{,}}{R_{,} \cdot \sin 1'' \cdot \cos l_{,,}} = \frac{s \cdot \sin A_2}{R_{,,} \cdot \sin 1'' \cdot \cos l_{,}}$$

$$= z_{,} \cdot \frac{\sin A_{,}}{\cos l_{,,}} = z_{,,} \cdot \frac{\sin A_{,,}}{\cos l_{,}}$$

from which we obtain the values

$$\omega = 4499''{\cdot}838 = 4500''{\cdot}355$$

having a difference $= 0''{\cdot}517$.